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REAL ANALYSIS LECTURE NOTES

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4. L^p SPACES AND THEIR RELATIVES

In this section we consider properties of function spaces, i.e., collections of functions defined on Euclidean domains that satisfy certain integrability or differentiability conditions. These are important examples of general spaces studied in functional analysis: Banach spaces, topological vector spaces, Fréchet spaces, etc. All integrals in this section will be with respect to the Lebesgue measure.

4.1. L^p **spaces.**

Definition 4.1. *For a domain $\Omega \subset \mathbb{R}^N$ and a real number p, $1 \le p < \infty$, a measurable function f is said to be of class $L^p(\Omega)$ if $\int_\Omega |f|^p < \infty$.*

Since $|f+g|^p \le 2^p(|f|^p + |g|^p)$ for all p, the space $L^p = L^p(\Omega)$ of all L^p-functions is a vector space. Define

$$||f||_p = \left(\int_\Omega |f|^p\right)^{1/p}. \tag{1}$$

Clearly, $||c\,f|| = |c|\,||f||$ for all $f \in L^p$ and $c \in \mathbb{R}$, and $||f|| = 0$ if and only if $f = 0$ a.e. on Ω. In Theorem 4.3 below we will show that $||f+g||_p \le ||f||_p + ||g||_p$. Thus, (1) defines a norm on L^p. Note that since $||f|| = 0$ only implies that f vanishes everywhere except a set of measure zero, one should understand elements of the space L^p as *equivalence classes* of functions satisfying Definition 4.1 with respect to the equivalence relation given by $f \sim g \Leftrightarrow f = g$ a.e.

For $p = \infty$ we define the space $L^\infty(\Omega)$ of bounded functions (more precisely *essentially bounded*) functions with the norm

$$||f||_\infty = \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| = \sup\left\{r \in \mathbb{R} : m(\{x : |f(x) - r| < \varepsilon\}) > 0 \text{ for all } \varepsilon > 0\right\}.$$

Theorem 4.2 (Hölder's inequality). *If $p, q \ge 1$ satisfy $1/p + 1/q = 1$, and $f \in L^p$, $g \in L^q$, then*

$$\int |fg| \le ||f||_p\ ||g||_q.$$

(If $p = 1$, we assume that $q = \infty$.)

Proof. We will leave the case $p = 1$, $q = \infty$ as an exercise for the reader, and assume that $p > 1$. We first establish so-called Young's inequality: for $a, b > 0$ and p and q as in Theorem 4.2, we have

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

To see that let $t = 1/p$, so $1 - t = 1/q$. Then, since log is a strictly concave function,

$$\log(ta^p + (1-t)b^q) \ge t\log(a^p) + (1-t)\log(b^q) = \log a + \log b = \log(ab),$$

from which the required inequality follows.

Now for the proof of Hölder's inequality, we may divide the functions f and g by their norms in the corresponding spaces, so we may assume that $||f||_p = ||g||_q = 1$. Using Young's inequality, we have

$$|f(x)g(x)| \leq \frac{|f(x)|^p}{p} + \frac{|g(x)|^q}{q}, \quad x \in \Omega.$$

Integrating the above inequality over Ω gives

$$||f(x)g(x)||_1 \leq 1/p + 1/q = 1,$$

which is what was needed to prove. □

The next theorem is essentially a corollary of Hölder's inequality.

Theorem 4.3 (Minkowski's inequality)**.** *For any* $p \geq 1$*,*

$$||f + g||_p \leq ||f||_p + ||g||_p.$$

Proof. When $p = 1$ or $p = \infty$ the inequality is trivial. For $1 < p < \infty$ we write

$$|f(x)+g(x)|^p = |f(x)+g(x)| \cdot |f(x)+g(x)|^{p-1} \leq |f(x)| \cdot |f(x)+g(x)|^{p-1} + |g(x)| \cdot |f(x)+g(x)|^{p-1}.$$

Integrating over Ω we obtain

$$||f + g||_p^p \leq \int |f| \cdot |f + g|^{p-1} + \int |g| \cdot |f + g|^{p-1}.$$

We now apply Hölder's inequality to both terms on the right. The first one yields

$$\int |f| \cdot |f + g|^{p-1} \leq ||f||_p \left| \int |f + g|^{(p-1)\frac{p}{p-1}} \right|^{\frac{p-1}{p}} \leq ||f||_p \cdot ||f + g||_p^{p-1}.$$

Similarly, the second term give

$$\int |g| \cdot |f + g|^{p-1} \leq ||g||_p \cdot ||f + g||_p^{p-1}.$$

Combining everything together yields

$$||f + g||_p^p \leq ||f||_p \cdot ||f + g||_p^{p-1} + ||g||_p \cdot ||f + g||_p^{p-1} = (||f||_p + ||g||_p) \cdot ||f + g||_p^{p-1},$$

From this the result follows. □

Another consequence of Hölder's inequality is the following. Let $m(\Omega) < +\infty$ and $f \in L^p(\Omega)$. Setting $g = 1$, we obtain

$$\| f \|_{L^1(\Omega)} \leq (m(\Omega))^{1/q} \| f \|_{L^p(\Omega)} . \tag{2}$$

Definition 4.4. *For* $p \geq 1$*, we say that a sequence* $\{f_n\} \subset L^p$ *converges to a function* $f \in L^p$ *in norm, if for every* $\varepsilon > 0$ *there exists* $N > 0$ *such that for all* $n > N$*, we have* $||f - f_n||_p < \varepsilon$*.*

If a series $\sum f_n$ of elements of L^p converges to an L^p-function we say that the series is *summable*. We call $\sum f_n$ *absolutely summable* if $\sum_n ||f_n|| < \infty$. (An absolutely convergent series of real numbers always converges, but this is not true in general when one considers series of elements of a normed space.)

Lemma 4.5. *A normed space* $(X, ||\cdot||)$ *is complete if and only if every absolutely summable series is summable.*

We leave the proof of the lemma as an exercise for the reader. Using the lemma we now can proof the following result.

Theorem 4.6 (Riesz-Fischer theorem). *for all $1 \le p < \infty$, the space L^p equipped with the norm $||\cdot||_p$ is a Banach space.*

Proof. We only need to prove that L^p is complete, i.e., that every Cauchy sequence in L^p converges in norm to an element of L^p. By Lemma 4.5, it suffices to show that every absolutely summable series is summable. Suppose that $\{f_n\}$ is such that $\sum ||f_n||_p = M < \infty$. Define

$$g_n(x) = \sum_{k=1}^{n} |f_k(x)|.$$

By Minkowski's inequality we have

$$||g_n||_p \le \sum_1^n ||f_k||_p \le M,$$

and so $\int g_n^p \le M^p$. For each x, the sequence $\{g_n(x)\}$ is an increasing sequence of extended real numbers (i.e., including the value ∞), and so it must converge to an extended real number $g(x)$. Then $g(x)$ is a measurable function, and since $g_n \ge 0$, $\int g^p \le M^p$ by Fatou's lemma. It follows that g^p is integrable, and so $g(x)$ is finite for a.e. x. For every x such that $g(x) < \infty$, the series $\sum_1^\infty f_k(x)$ converges absolutely, so in particular, it converges to a real number $s(x)$. We set $s(x) = 0$ for those x where $g(x) = \infty$. Thus we constructed a function $s(x)$ which is the limit a.e. of partial sums $s_n = \sum_1^n f_k$. It follows that s is measurable, and since $|s_n(x)| \le |g(x)|$, we have $s(x) \le g(x)$. Consequently, $s \in L^p$, and

$$|s_n(x) - s(x)|^p \le 2^p[g(x)]^p.$$

Since $2^p g^p$ is integrable, and $|s_n(x) - s(x)|^p \to 0$, we have by the Lebesgue Convergence theorem,

$$\int |s_n - s|^p \to 0.$$

Therefore, $||s_n - s||^p \to 0$, which proves the theorem. □

Let us now describe some natural dense subsets of $L^p(\mathbb{R}^n)$.

Proposition 4.7. *Let Ω be a bounded measurable subset of $\mathbb{R}^n$. Then the set of all functions continuous on $\mathbb{R}^n$ is dense in $L^1(\Omega)$.*

Proof. It follows from the definition of the Lebesgue integral that the space of integrable simple functions on X is dense in $L^1(\Omega)$. Furthermore, every simple function is a linear combination of characteristic functions of some measurable subsets of Ω. Hence it suffices to show that the characteristic function χ_Y of a set Y of finite measure is a limit in L^1 of a sequence of continuous functions. From the definition of the Lebesgue measure for any given $\varepsilon > 0$ there exists an open subset $G \supset Y$ in $\mathbb{R}^n$ such that $m(G \backslash Y) < \varepsilon/2$. This also implies that there exists a closed subset $F \subset Y$ in $\mathbb{R}^n$ such that $m(G) - m(F) < \varepsilon$. Consider now the function

$$\varphi_\varepsilon(x) = \frac{d(x, G^c)}{d(x, G^c) + d(x, F)},$$

where d is the Euclidean distance from x to the set which is the second argument of the function. The function φ_ε is continuous on $\mathbb{R}^n$ since the denominator is strictly positive. Furthermore, φ_ε vanishes on G^c, the complement of G, and is identically equal to 1 on F. Hence,

$$\int_{\mathbb{R}^n} |\chi_Y - \varphi_\varepsilon| \le \varepsilon$$

which proves the proposition. □

This easily implies that $L^1(\Omega)$ is a separable space. Indeed, the space of polynomials with rational coefficients is dense in the space of continuous functions (the Weierstrass theorem). One can also show that $L^p(\Omega)$ is separable for $1 \leq p < \infty$. It is also easy to see that this remains true even if $m(\Omega) = +\infty$.

4.2. **Topological vector spaces and their duals.**

Definition 4.8. *If a vector space X (over the field of reals) is equipped with some topology, we called X a topological vector space if the map $X \times X \to X$ corresponding to vector addition in X and the map $\mathbb{R} \times X \to X$ corresponding to scalar multiplication are both continuous.*

Sometimes it is required that the topology on X is Hausdorff. This is always the case if the topology comes from a metric (in particular, from a norm) on X.

Example 4.1. We give some examples of topological vector spaces.

(i) The space L^p is a topological vector space for any $p \geq 1$ (prove it!).

(ii) The space $C[0,1]$ of continuous functions on the interval $[0,1]$. One can show that this is a Banach space equipped with the norm $||f|| = \sup_{x\in[0,1]} |f(x)|$. In fact, any normed space, complete or not, is a topological vector space.

(iii) The next is an example of a topological vector space which is not a normed space. Consider the space $C^\infty([0,1])$ of smooth functions on $[0,1]$. The topology on $C^\infty([0,1])$ can be described as follows. For every integer $k \geq 0$ we define a *semi-norm*

$$\|f\|_k = \sup_{x\in[0,1]} \{|f^{(k)}(x)| : x \in [0,1]\}.$$

Here $f^{(k)}$ is the derivative of f of order k. That $||\cdot||_k$ is a semi-norm, rather than a norm, means that $||f||_k = 0$ may hold for nonzero functions, for example any constant c satisfies $||c||_1 = 0$. The space $C^\infty([0,1])$ a complete metric space with the metric given by

$$d(f,g) = \sum_{k=0}^{\infty} 2^{-k} \frac{\|f-g\|_k}{1+\|f-g\|_k} \qquad f,g \in C^\infty([0,1]). \tag{3}$$

Topological vector spaces equipped with a complete metric that comes from a countable collection of semi-norms are called *Fréchet spaces.*

◇

Recall that a *functional* over a vector space X is a linear map $\phi : X \to \mathbb{R}$.

Definition 4.9. *Given a topological vector space X, the space of continuous linear functionals is called the dual space of X, and denoted by X^*.*

One can show that if a topological vector space X is finite-dimensional, then every linear map on X is continuous. For infinite-dimensional vector spaces continuity of a functional is a nontrivial condition.

Example 4.2.

(i) For a space L^p, $p \geq 1$, choose some $g \in L^q$ with $1/p + 1/q = 1$. Then the map $\phi_g : L^p \to \mathbb{R}$, given by

$$\phi_g(f) = \langle \phi_g, f \rangle = \int fg, \tag{4}$$

is a continuous linear functional on L^p.

(ii) On $C[-1,1]$ the map I given by $I(f) = \int_{-1}^{1} f dx$ is a continuous linear functional. Another example of a continuous linear functional on $C[0,1]$ is given by $\delta_0(f) = f(0)$. This is the famous Dirac delta-function.

(iii) On $C^\infty[-1,1]$ consider the map $\Phi : C^\infty[-1,1] \to \mathbb{R}$ given by

$$\langle \Phi, f \rangle = \delta_0(f) + \frac{\partial f}{\partial x}(0).$$

Continuity of this functional can be verified from the convergence on the space $C^\infty[-1,1]$ given by the metric in (3).

Lemma 4.10. *Let X be a topological vector space, and ϕ be a linear functional. Then $\phi \in X^*$ if and only if ϕ is continuous at some point $x \in X$.*

Proof. For the proof in the nontrivial direction, let $y \in X$ be arbitrary. For a given $\varepsilon > 0$, choose a neighbourhood U of x such that

$$|\phi(x) - \phi(x')| < \varepsilon, \quad \text{for all } x' \in U.$$

Then the set $V = U + (y - x)$ is a neighbourhood of y. If $z \in V$, then $z + x - y \in U$, and so

$$|\phi(z) - \phi(y)| = |\phi(z - y + x) - \phi(x)| < \varepsilon,$$

which shows continuity of ϕ at y. $\square$

Thus, it suffices to test continuity of a functional at one point, for example, at the origin. Using this, one can show that continuity of a functional on a topological vector space is equivalent to its boundedness in some neighbourhood of the origin.

The dual space is itself a topological vector space. Its topology can be defined as follows. We say that a sequence $\phi_n \in X^*$ converges to $\phi \in X^*$ if for any $x \in X$, we have $\lim \phi_n(f) = \phi(f)$. This is the so-called weak* topology. It is the weakest topology that makes the pairing $X \times X^* \to \mathbb{R}$ a continuous operation.

For a normed space X its dual space is also normed. The norm on X^* is given by

$$||\phi||_* = \sup_{f \in X \setminus \{0\}} \frac{|\phi(f)|}{\|f\|}.$$

For example, if $g \in L^q$, then the functional ϕ_g given by (4) has the norm which is equal to that of g: $||\phi_g||_* = ||g||_q$. Indeed, by Hölder's inequality,

$$\left| \int fg \right| \leq \int |fg| \leq ||f||_p \, ||g||_q,$$

which shows that $||\phi_g||_* \leq ||g||_q$. On the other hand, if $f = |g|^{q/p} \operatorname{sgn} g$, then $|f|^p = |g|^q = f\, g$, and so $||f||_p = ||g||_q^{q/p}$. Therefore,

$$\langle g, f \rangle = \int f\, g = \int |g|^q = (||g||_q)^q = ||g||_q \, ||f||_p,$$

which proves our assertion. In fact, the following holds.

Theorem 4.11 (Riesz Representation theorem)**.** *A linear map $\phi : L^p \to \mathbb{R}$ is a continuous linear function on L^p if and only if there exists $g \in L^q$, $1/p + 1/q = 1$, such that for all $f \in L^p$,*

$$\langle \phi, f \rangle = \int g\, f.$$

We do not give the proof of this theorem. Observe that now we have two topologies on the space L^q: the normed topology and the weak* topology. These two are not the same, with weak* being weaker than the normed topology (sometimes called the *strong* topology), i.e., weak* topology has fewer opens sets. To see this it is enough to construct an example of a sequence in L^q that converges weakly, but not in norm. Consider the space $L^2(0,1)$ which is the dual of itself, and consider the sequence

$$\phi_n(x) = \begin{cases} \sqrt{n}, \ x \in (0, \frac{1}{n}), \\ 0, \ x \in [\frac{1}{n}, 1) \end{cases}.$$

Clearly, $\phi_n \in L^2(0,1)$, and $\phi_n(x) \to 0$ point-wise on $(0,1)$ as $n \to \infty$. We use the following fact: if a sequence ϕ_n converges to ϕ pointwise and converges to a function ϕ' in norm, then $\phi = \phi'$ (prove it!). From this it follows that the sequence ϕ_n does not converge in norm, since for any n we have

$$||\phi_n - 0||_2 = \left(\int_0^1 \phi_n^2 \right)^{1/2} = \left(\int_0^{1/n} n \right)^{1/2} = 1.$$

But ϕ_n converges to 0 in weak* topology because for any $f \in L^2(0,1)$,

$$\langle \phi_n, f \rangle = \int_0^1 \phi_n f = \int_0^{1/n} \sqrt{n}\, f \le \left(\int_0^{1/n} f \right)^{1/2},$$

by Hölder's inequality. As $n \to \infty$ the right hand-side in the above formula converges to zero for every $f \in L^2$, which gives weak convergence $\phi_n \to 0$. On the other hand, by Hölder's inequality, convergence in norm always implies convergence in weak* topology.

4.3. **Product spaces and Fubini's theorem.** Let $(\mathbb{R}^N, m)$ and $(\mathbb{R}^n, \mu)$ be the Euclidean spaces with the corresponding Lebesgue measures. Then on the space $\mathbb{R}^N \times \mathbb{R}^n = \mathbb{R}^{N+n}$ we may define a new measure λ as follows: If $A \subset \mathbb{R}^N$ and $B \subset \mathbb{R}^n$ are measurable, then

$$\lambda(A \times B) := m(A) \cdot \mu(B).$$

One can show that λ can be extended from the collection of product sets in $\mathbb{R}^{N+n}$ to a wide class of sets in $\mathbb{R}^{N+n}$. In fact, one can prove that the measure λ obtained this way is nothing but the Lebesgue measure on $\mathbb{R}^{N+n}$.

The following theorem gives sufficient conditions that allows one to replace an integral with respect to a product measure by an iterated integral. We use dx (resp. dy) to denote integration with respect to measure m (resp. μ), and $dx\,dy$ to denote integration with respect to λ.

Theorem 4.12 (Fubini's theorem)**.** *Let $X \times Y \subset (\mathbb{R}^N, m) \times (\mathbb{R}^n, \mu)$ be measurable, and $f \in L^1(X \times Y, \lambda)$. Then*

(i) *for a.e. x, the function $f_x(y) = f(x,y)$ is integrable on Y;*
(ii) *for a.e. y, the function $f_y(x) = f(x,y)$ is integrable on X;*
(iii) *the function $x \to \int_Y f_x(y)dy$ is integrable on X;*
(iv) *the function $y \to \int_X f_y(x)dx$ is integrable on Y;*
(v) $\int_X \left[\int_Y f_x(y)dy\right] dx = \int_{X\times Y} f(x,y)dx\,dy = \int_Y \left[\int_X f_y(x)dx\right] dy.$

For the proof of Fubini's theorem one can reduce the problem to simple functions and then use the Lebesgue convergence theorem. We omit the details. Below we state a variation of Fubini's theorem that does not require integrability of the function f.

Theorem 4.13 (Tonelli's theorem)**.** *Let $X \times Y \subset (\mathbb{R}^N, m) \times (\mathbb{R}^n, \mu)$ be measurable, and $f : X \times Y \to \mathbb{R}$ be a nonnegative measurable function. Then*

(i) *for a.e.* x*, the function* $f_x(y) = f(x, y)$ *is measurable on* Y*;*
(ii) *for a.e.* y*, the function* $f_y(x) = f(x, y)$ *is measurable on* X*;*
(iii) *the function* $x \to \int_Y f_x(y)dy$ *is measurable on* X*;*
(iv) *the function* $y \to \int_X f_y(x)dx$ *is measurable on* Y*;*
(v) $\int_X \left[\int_Y f_x(y)dy\right] dx = \int_{X\times Y} f(x, y)dx\, dy = \int_Y \left[\int_X f_y(x)dx\right] dy.$

In the remaining part of the subsection we discuss another property of product spaces: possibility of interchanging integration and differentiation.

Lemma 4.14. *Let* $X \times Y \subset \mathbb{R}^N \times \mathbb{R}^n$ *be a product of open subsets, and let* $m(X) < \infty$*. Suppose* $f(x, y) : X \times Y \to \mathbb{R}$ *is uniformly continuous on the closure of* $X \times Y$*. Then the function*

$$F(y) = \int_X f(x, y)dx \tag{5}$$

is continuous on Y*.*

Proof. Uniform continuity of f on $X \times Y$ means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x, y) - f(x', y')| < \varepsilon \quad \Longleftarrow \quad |(x, y) - (x', y')| < \delta.$$

Then,

$$|F(y) - F(y')| = \left|\int_X f(x, y) - f(x, y')dx\right| \le \int_X |f(x, y) - f(x, y')|dx \le \varepsilon m(X),$$

which proves continuity of F. □

The next theorem gives a sufficient condition under which we can differentiate under the integral sign. It can also be interpreted as commutativity of the operations of integration and differentiation under the given assumptions.

Theorem 4.15. *Let* X, Y *be as in the previous lemma. Assume that for some* $1 \le i \le m$*, the functions* $f(x, y)$ *and* $\frac{\partial f}{\partial y_i}$ *are uniformly continuous on the closure of* $X \times Y$*. Then the function* $F(y)$ *defined by* (5) *is of class* $C^1(Y)$*, and*

$$\frac{\partial F}{\partial y_i}(y) = \int_X \frac{\partial f}{\partial y_i}(x, y)dx.$$

Proof. Let $y_0 \in Y$ be arbitrary. For $h = (0, \ldots, 0, h_i, 0, \ldots, 0) \in \mathbb{R}^m$ by the Mean Value theorem we have

$$\frac{F(y_0 + h) - F(y_0)}{h_i} = \int_X \frac{f(x, y_0 + h) - f(x, y_0)}{h_i} dx = \int_X \frac{\partial f}{\partial y_i}(x, y_0 + \theta h)dx, \quad \theta \in (0, 1).$$

Note that θ depends on x. Since $\frac{\partial f}{\partial y_i}$ is uniformly continuous on $X \times Y$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left|\frac{\partial f}{\partial y_i}(x, y) - \frac{\partial f}{\partial y_i}(x, y_0)\right| < \varepsilon \quad \Longleftarrow \quad |y - y_0| < \varepsilon.$$

Thus, for $|h| = |h_i| < \delta$, we have

$$\left|\frac{F(y_0 + h) - F(y_0)}{h_i} - \int_X \frac{\partial f}{\partial y_i}(x, y_0)dx\right| = \left|\int_X \left(\frac{\partial f}{\partial y_i}(x, y_0 + \theta h) - \frac{\partial f}{\partial y_i}(x, y_0)\right) dx\right| \le$$
$$\int_X \left|\frac{\partial f}{\partial y_i}(x, y_0 + \theta h) - \frac{\partial f}{\partial y_i}(x, y_0)\right| dx \le \varepsilon m(X).$$

This shows that

$$\lim_{h_i \to 0} \frac{F(y_0 + h) - F(y_0)}{h_i} = \int_X \frac{\partial f}{\partial y_i}(x, y_0)dx.$$

Finally, the continuity of $\frac{\partial F}{\partial y_i}$ follows from the lemma. $\square$