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REAL ANALYSIS LECTURE NOTES

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5. Divergence theorem and consequences

5.1. **Integration on hypersurfaces.** By a hypersurface Γ of class C^k in $\mathbb{R}^n$, $k \in \mathbb{Z}^+$, we mean a compact subset of $\mathbb{R}^n$ admitting a finite covering by open connected subsets U_j: $\Gamma \subset \cup_{j=1}^N U_j$, with the following property: For every j there exists a function $\rho_j \in C^k(U_j)$ such that the gradient $\nabla\rho_j(x) = (\frac{\partial\rho_j}{\partial x_1}, ..., \frac{\partial\rho_j}{\partial x_n})$ does not vanish in U_j and $\Gamma \cap U_j = \{x \in U_j : \rho_j(x) = 0\}$. Such a function ρ_j is called *a local defining function* of Γ. Very often we will deal with the case when $\Gamma = \{x \in \mathbb{R}^n : \rho(x) = 0,\ \nabla\rho \neq 0\}$, where ρ is a C^k-function on $\mathbb{R}^n$.

Another way to define a hypersurface is through parametrization. Let D be an open connected subset of $\mathbb{R}^{n-1}$ and $\Phi = (\Phi_1, ..., \Phi_n) : D \to \mathbb{R}^n$ be an injective map of class $C^k(D)$. The hypersurface $\Gamma = \Phi(D)$ is called a *parametrized hypersurface*.

Example 5.1. On $\mathbb{R}^2$ with coordinates (x, y) consider for some $k \in \mathbb{Z}^+$

$$\rho(x,y) = \begin{cases} y, & \text{for } x \le 0, \\ y - x^k, & \text{for } x > 0 \end{cases} .$$

Then $\Gamma \subset \mathbb{R}^2$ given by $\rho(x, y) = 0$ is a hypersurface of class C^{k-1}. It admits a global C^{k-1}-smooth parametrization $\Phi : \mathbb{R} \to \Gamma$ given by $x \mapsto (x, x^k)$ for $x > 0$, and $x \mapsto (x, 0)$ for $x \le 0$. $\diamond$

Let now Ω be a bounded domain (an open connected subset) of $\mathbb{R}^n$ with the boundary $\partial\Omega$ consisting of a finite number of disjoint hypersurfaces Γ_k of class C^1. A local defining function ρ_j for Γ_k as defined above is called *a local defining function of* Ω if $\Omega \cap U_j = \{x \in U_j : \rho_j(x) < 0\}$. Then the gradient vector $\nabla\rho_j(x)$ defines the outward-pointing normal direction to $\partial\Omega$ at a point $x \in \partial\Omega$. We denote by

$$\vec{n}(x) := \frac{\nabla\rho(x)}{|\nabla\rho(x)|}$$

the unit vector in the outward-pointing normal direction. Let $p = (p_1, ..., p_n)$ be a boundary point of Ω and ρ be a local defining function of $\partial\Omega$ near p. Since $\nabla\rho(p) \neq 0$, then $\partial\rho(p)/\partial x_k \neq 0$ for some $1 \le k \le n$. By the Implicit Function theorem there exists a neighbourhood U of p, a function ψ of class C^1 such that

$$\partial\Omega \cap U = \{x \in U : x_k = \psi(x_1, ..., x_{k-1}, x_{k+1}, ..., x_n)\}. \tag{1}$$

Shrinking U if necessary we may assume that $U = U' \times U''$, where U' is a ball in the space $\mathbb{R}^{n-1}$ centred at $(p_1, ..., p_{k-1}, p_{k+1}, ..., p_n)$ and U'' is an interval in $\mathbb{R}$ centred at p_k. This representation allows us to view $(x_1, ..., x_{k-1}, x_{k+1}, ..., x_n)$ as local coordinates on $\partial\Omega$: the projection

$$\pi_k : x \mapsto (x_1, ..., x_{k-1}, x_{k+1}, ..., x_n)$$

$$\pi_k : \partial\Omega \cap U \longrightarrow U'$$

is bijective. We point out that

$$\pi_k^{-1}(x_1, ..., x_{k-1}, x_{k+1}, ..., x_n) = (x_1, ..., x_{k-1}, \psi(x_1, ..., x_{k-1}, x_{k+1}, ..., x_n), x_{k+1}, ..., x_n)$$

for $(x_1, ..., x_{k-1}, x_{k+1}, ..., x_n) \in U'$. The map $\pi_k^{-1} : U' \to \partial\Omega \cap U$ is clearly a local parametrization of the hypersurface $\partial\Omega$. We call U a *coordinate neighbourhood* of p.

Example 5.2. The upper hemisphere $S^+ = S^2 \cap \{z > 0\}$ in $\mathbb{R}^3$ with coordinates (x, y, z) is the graph of the function $z = \sqrt{1 - x^2 - y^2}$. The unit normal vector to S^+ at a point $(x, y, z) \in S^+$ is $\vec{n} = (x, y, z)$. $\diamond$

Now let f be a continuous (this assumption can be considerably weakened) function on $\partial\Omega$. Our goal is to define the integral of f over $\partial\Omega$ as a *surface integral* . If an open set $X \subset \partial\Omega$ admits a parametrization $\Phi : D \to \mathbb{R}^n$, $\Phi(D) = X \subset \partial\Omega$ then we define

$$\int_X f(x)dS = \int_D f \circ \Phi(t)|\vec{N}|dt, \tag{2}$$

where the coordinates of the vector $\vec{N}$ are determined from $\vec{N} = |\det(\nabla\Phi_1, ..., \nabla\Phi_n, \vec{e})|$. Here $\vec{e} = (\vec{e}_1, \dots, \vec{e}_n)$ is a formal vector whose coordinates are the vectors of the standard basis in $\mathbb{R}^n$. In fact, one can show that $\vec{N}$ is the normal vector to $X \subset \partial\Omega$.

Now if U is a coordinate neighbourhood where $\partial\Omega$ admits representation as in (1) and X is an open subset in $\partial\Omega \cap U$ then

$$\int_X f dS = \int_{\pi_k(X)} f \circ \pi_k^{-1}(1 + \| \nabla\psi \|^2)^{1/2} dx_1...dx_{k-1}dx_{k+1}...dx_n. \tag{3}$$

Both definitions agree because $(1 + \| \nabla\psi \|^2)^{1/2}$ is just the length of the normal vector

$$\vec{N} = \left(\frac{\partial\psi}{\partial x_1}, ..., 1, ..., \frac{\partial\psi}{\partial x_n}\right), \tag{4}$$

(here 1 is on the k-th position) corresponding to the local parametrization of $\partial\Omega$. We refer to dS or the equivalent expression in a local parametrization as the *hypersurface area measure* (or the *element of the surface area* in some literature). Let ν_k be the angle between $\vec{n} = \vec{N} / \| \vec{N} \|$ and the vector $\vec{e}_k$ (the k-th vector of the standard base of $\mathbb{R}^n$). Then

$$\cos\nu_k = (\vec{e}_k, \vec{n}) = (1 + \| \nabla\psi \|^2)^{-1/2}.$$

Thus,

$$\int_X f dS = \int_{\pi_k(X)} f \circ \pi_k^{-1} \frac{1}{\cos\nu_k} dx_1...dx_{k-1}dx_{k+1}...dx_n. \tag{5}$$

If $f \equiv 1$, then the integral $\int_X dS$ represents the area of X. This terminology comes from $\mathbb{R}^3$, where the integral is indeed the area of a surface, while for $n > 3$, it is actually the $(n-1)$-dimensional volume.

Example 5.3. Consider the surface integral of a continuous function $f(x, y, z)$ over the upper hemisphere $S^+ = S^2 \cap \{z > 0\} \subset \mathbb{R}^3$. First we use the parametrization $z = \psi(x, y) = \sqrt{1 - x^2 - y^2}$ for $x^2 + y^2 < 1$. Then

$$|1 + |\nabla\psi|^2| = 1 + \frac{x^2}{1 - x^2 - y^2} + \frac{y^2}{1 - x^2 - y^2} = \frac{1}{1 - x^2 - y^2}.$$

Therefore, from (3) we obtain

$$\int_{S^+} f dS = \int_{\{x^2 + y^2 < 1\}} f(x, y, \sqrt{1 - x^2 - y^2}) \frac{dxdy}{\sqrt{1 - x^2 - y^2}}.$$

Now we use the parametrization of S^+ that comes from the spherical coordinates. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$\Phi(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

Then $\Phi\left((0, \pi/2) \times (0, 2\pi)\right) = S^+$ (excluding a set of measure 0). To apply (2) we first compute vectors of partial derivatives with respect to θ and ϕ. We have

$$\Phi_\theta = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta), \quad \Phi_\phi = (-\sin\theta\sin\phi, \sin\theta\cos\phi, 0).$$

Then

$$\vec{N} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{vmatrix} = \sin\theta(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta),$$

and so $|\vec{N}| = \sin\theta$. We conclude that

$$\int_{S^+} f dS = \int_{(0,\pi/2)\times(0,2\pi)} f(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\sin\theta \; d\theta \, d\phi.$$

That both integrals agree can be verified, for example, by calculating the surface area of S^+ using these two representations of the surface integral. $\diamond$

Finally, if U_j is an open covering of $\partial\Omega$ by coordinate neighbourhoods, we set $X_k = U_k \setminus \cup_{j=1}^{k-1} U_j$ so that $\partial\Omega = \cup X_k$ and X_k are disjoint. Then we set

$$\int_{\partial\Omega} f dS = \sum_k \int_{X_k} f dS.$$

One can view this as a definition of the surface integral over $\partial\Omega$. It is not difficult to verify that the integral is well-defined, i.e., it is independent of the choice of the covering by coordinate neighbourhoods, local defining functions, etc. We leave this verification as an exercise for the reader.

5.2. **Divergence theorem.** The following theorem connects the integral over a domain Ω with the surface integral over its boundary Ω. It was discussed in some form in the work of Lagrange, Gauss, and most notably Ostrogradski, who gave a proof that would be considered complete by modern standards. It is sometimes referred to as Gauss-Ostrogradski theorem.

Recall that a vector field F on a domain $\Omega \subset \mathbb{R}^n$ is simply a map $F : \Omega \to \mathbb{R}^n$. The geometric interpretation of a vector field (which becomes nontrivial and important when one considers abstract manifolds) is that at each point $x \in \Omega$ the value $F(x)$ is thought of as a vector in $\mathbb{R}^n$ originating at x. For example, given a function $f : \Omega \to \mathbb{R}$, the gradient ∇f is a vector field on Ω. Another example is a vector field given by (4) assigning to every boundary point of $\partial\Omega$ a normal vector $\vec{N}$ to Ω. The divergence of a vector field F is defined as $div\vec{F} = \frac{\partial F_1}{\partial x_1} + ... + \frac{\partial F_n}{\partial x_n}$.

Theorem 5.1 (Divergence theorem)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$ with the boundary of class C^1. Let $\vec{F} = (F_1, ..., F_n)$ be a a vector field of class $C(\overline{\Omega}) \cap C^1(\Omega)$. . Then*

$$\int_\Omega div\vec{F} dx = \int_{\partial\Omega} (\vec{F}, \vec{n}) dS, \tag{6}$$

where (a, b) denotes the usual scalar product of two vectors in $\mathbb{R}^n$ and $\vec{n}$ denotes the vector field of the outward-pointing unit normals to $\partial\Omega$.

For $n = 1$ the Divergence theorem becomes the Fundamental Theorem of Calculus.

Proof. For simplicity of notation we assume that $n = 3$, the proof in the general case is completely analogous. We will assume that $\Omega = \{(x, y, z) : (x, y) \in D, \ \psi_1(x, y) < z < \psi_2(x, y)\}$, where D is a domain in $\mathbb{R}^2$, and ψ and ϕ are smooth functions on D. Moreover, we assume that a similar representation is also valid for projections onto the other two coordinate planes. Such domains sometimes are called *simple*. If the domain Ω is not simple in all three directions, then we may divide it into smaller domains Ω_i which are simple. Adding the results for each i gives the Divergence theorem for Ω and $\partial\Omega$. Indeed, since after splitting Ω the surface integrals over the newly introduced boundaries occur twice with the opposite normal vectors $\vec{n}$, their sum is equal to zero, and we end up with the surface integral over the original $\partial\Omega$.

Denote by Γ_j the surface

$$\Gamma_j = \{(x, y, \psi_j(x, y)) \in \mathbb{R}^3 : (x, y) \in D\}, \quad j = 1, 2.$$

Then, by Fubini's theorem,

$$\int_\Omega \frac{\partial F_3(x, y, z)}{\partial z} dxdydz = \int_D \left(\int_{\psi_1(x,y)}^{\psi_2(x,y)} \frac{\partial F_3(x, y, z)}{\partial z} dz \right) dxdy =$$

$$\int_D F_3(x, y, \psi_2(x, y))dxdy - \int_D F_3(x, y, \psi_1(x, y))dxdy = \int_{\Gamma_1 \cup \Gamma_2} F_3(x, y, z) \cos \nu_3 dS,$$

where ν_3 is the angle between the vector $\vec{e}_3$ and the normals $(\frac{\partial \psi_2}{\partial x}, \frac{\partial \psi_2}{\partial y}, 1)$ when $(x, y, z) \in \Gamma_2$ and $(-\frac{\partial \psi_1}{\partial x}, -\frac{\partial \psi_2}{\partial y}, -1)$ when $(x, y, z) \in \Gamma_1$ respectively. In the last step we used (5).

Let now $\Gamma_3 = \{(x, y, z) : (x, y) \in \partial D, \psi_1(x, y) < z < \psi_2(x, y)\}$ be the "vertical" part of $\partial\Omega$. Let ν_3 still denote the angle between Γ_3 and the outward-pointing unit normal vector to Γ_3. Then $\nu_3 = \pi/2$ and $\cos \nu_3 = 0$. Then

$$\int_{\Gamma_3} F_3(x, y, z) \cos \nu_3 dS = 0.$$

Since $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ we can write

$$\int_{\partial\Omega} F_3(x, y, z) \cos \nu_3 dS = \int_{\Gamma_1 \cup \Gamma_2} F_3(x, y, z) \cos \nu_3 dS,$$

and therefore,

$$\int_\Omega \frac{\partial F_3(x, y, z)}{\partial z} dxdydz = \int_{\partial\Omega} F_3(x, y, z) \cos \nu_3 dS. \tag{7}$$

Similarly, we establish the formulas

$$\int_\Omega \frac{\partial F_1(x, y, z)}{\partial x} dxdydz = \int_{\partial\Omega} F_1(x, y, z) \cos \nu_1 dS, \tag{8}$$

and

$$\int_\Omega \frac{\partial F_2(x, y, z)}{\partial y} dxdydz = \int_{\partial\Omega} F_2(x, y, z) \cos \nu_2 dS, \tag{9}$$

where ν_1 and ν_2 are angles between the outward-pointing unit normal to $\partial\Omega$ and the standard base vectors $\vec{e}_1$ and $\vec{e}_2$. Taking the sum in (7), (8), (9) we obtain

$$\int_\Omega div \vec{F} dx = \int_{\partial\Omega} (F_1(x, y, z) \cos \nu_1 + F_2(x, y, z) \cos \nu_2 + F_3(x, y, z) \cos \nu_3, \tag{10}$$

which is precisely (6) for dimension 3. $\square$

In conclusion we mention some useful consequence of the Divergence theorem. Let f be a function of class $C(\overline{\Omega}) \cap C^1(\Omega)$. Applying the Divergence theorem to the vector field $\vec{F} = f\vec{e}_k$ we obtain

$$\int_\Omega \frac{\partial f}{\partial x_k}(x)dx = \int_{\partial\Omega} f(x)(\vec{e}_k, \vec{n}(x))dS. \tag{11}$$

Let $f = u \cdot v$. Then, since

$$\frac{\partial u}{\partial x_k} v = \frac{\partial(uv)}{\partial x_k} - u\frac{\partial v}{\partial x_k},$$

formula (11) gives

$$\int_\Omega \frac{\partial u}{\partial x_k}(x)v(x)dx = \int_{\partial\Omega} u(x)v(x)(\vec{e}_k, \vec{n}(x))dS - \int_\Omega u\frac{\partial v}{\partial x_k}dx, \tag{12}$$

which is just the multidimensional *integration by parts* formula. Since $(\vec{e}_k, \vec{n}) = \cos\nu_k$, where ν_k is the angle between vectors e_k and $\vec{n}$, the integration by parts formula can be rewritten as

$$\int_\Omega \frac{\partial u}{\partial x_k}(x)v(x)dx = \int_{\partial\Omega} u(x)v(x)\cos\nu_k dS - \int_\Omega u\frac{\partial v}{\partial x_k}dx.$$

Recall that the Laplacian of a C^2-smooth function $u(x_1, \ldots, x_n)$ is the function $\Delta u = \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2}(x)$. Consider now two functions u and v of class $C^2(\Omega) \cap C^1(\overline{\Omega})$ such that their Laplacians Δu and Δv are integrable in Ω. Clearly,

$$\Delta u = div(\nabla u).$$

Furthermore, for every boundary point $x \in \partial\Omega$ the scalar product $(\nabla u(x), \vec{n}(x))$ coincides with the directional derivative $\frac{\partial u}{\partial \vec{n}}$. On the other hand,

$$v\Delta u = v div(\nabla u) = div(v\nabla u) - (\nabla u, \nabla v).$$

Integrating this identity over Ω and applying the Divergence theorem we obtain *the first Green's formula*:

$$\int_\Omega v\Delta u dx = \int_{\partial\Omega} v\frac{\partial u}{\partial \vec{n}}dS - \int_\Omega (\nabla u, \nabla v)dx. \tag{13}$$

Similarly we have

$$\int_\Omega u\Delta v dx = \int_{\partial\Omega} u\frac{\partial v}{\partial \vec{n}}dS - \int_\Omega (\nabla v, \nabla u)dx.$$

Subtracting this last equality from (13) we obtain *the second Green's formula*

$$\int_\Omega (v\Delta u - u\Delta v)dx = \int_{\partial\Omega} \left(v\frac{\partial u}{\partial \vec{n}} - u\frac{\partial v}{\partial \vec{n}}\right) dS. \tag{14}$$

5.3. **Change of variables in the integral.** The following theorem is the multidimensional version of the substitution rule in the integral.

Theorem 5.2. *Let $\Phi : \bar{\Omega}' \to \bar{\Omega}$ be a C^1-diffeomorphism between two domains in $\mathbb{R}^n$ with C^1-smooth boundary, and let $f \in L^1(\Omega)$. Then*

$$\int_\Omega f(x)dx = \int_{\Omega'} f \circ \Phi(y)|J_\Phi(y)|dy,$$

where $|J_\Phi|$ denotes the determinant of the differential (Jacobian matrix) of the map Φ.

Proof. We will give the proof for $n = 3$. Assume that the coordinates in Ω are (x, y, z) and (u, v, w) in Ω'. Suppose $\partial\Omega$ is parametrized by a function

$$\phi(s,t) \to (x(s,t), y(s,t), z(s,t)), \quad (s,t) \in D \subset \mathbb{R}^2.$$

Then after substituting ϕ into Φ, the hypersurface $\partial\Omega'$ is given by some function

$$(s,t) \to (u(s,t), v(s,t), w(s,t)).$$

Define the function

$$F(x,y,z) = \int_0^z f(x,y,\xi)d\xi.$$

Then $\frac{\partial F}{\partial z} = f$ in Ω. We will use the notation $\frac{\partial(x,y)}{\partial(s,t)}$ to denote the determinant of the Jacobian matrix obtained by taking partial derivatives of the functions $x(s,t)$ and $y(s,t)$ with respect to variables (s,t). Similar notation will be used for any other collection of functions and variables. Set

$$\vec{N} = (N_1, N_2, N_3) = \left(\frac{\partial(y,z)}{\partial(s,t)}, \frac{\partial(z,x)}{\partial(s,t)}, \frac{\partial(x,y)}{\partial(s,t)} \right),$$

and

$$\vec{N}' = (N_1', N_2', N_3') = \left(\frac{\partial(v,w)}{\partial(s,t)}, \frac{\partial(w,u)}{\partial(s,t)}, \frac{\partial(u,v)}{\partial(s,t)} \right).$$

We claim that

$$N_3 = \frac{\partial(\Phi_1,\Phi_2)}{\partial(v,w)} N_1' + \frac{\partial(\Phi_1,\Phi_2)}{\partial(w,u)} N_2' + \frac{\partial(\Phi_1,\Phi_2)}{\partial(u,v)} N_3'. \tag{15}$$

This can be verified by writing

$$x = \Phi_1(u(s,t), v(s,t), w(s,t)), \quad y = \Phi_2(u(s,t), v(s,t), w(s,t)),$$

differentiating these equations with respect to s and t and then substituting into $N_3 = x_s y_t - x_t y_s$. The vectors $\vec{N}$ and $\vec{N}'$ are normal to $\partial\Omega$ and $\partial\Omega'$ respectively, say, N is the outward-pointing normal to $\partial\Omega$, and $\vec{N}'$ is the inward-point normal to $\partial\Omega'$. Then

$$\vec{n} = \vec{N}/|\vec{N}| \ \text{ and } \ \vec{n}' = -\vec{N}'/|\vec{N}'| \tag{16}$$

are the corresponding unit normal vectors. By the Divergence theorem,

$$\int_\Omega f\, dx\, dy\, dz = \int_\Omega \frac{\partial F}{\partial z} dx\, dy\, dz = \int_{\partial\Omega} F \cos(e_3, \vec{n}) dS = \int_D F N_3\, ds\, dt.$$

Substitution of N_3 from (15) gives

$$\int_\Omega f\, dx\, dy\, dz = \int_D F \left(\frac{\partial(\Phi_1,\Phi_2)}{\partial(v,w)} N_1' + \frac{\partial(\Phi_1,\Phi_2)}{\partial(w,u)} N_2' + \frac{\partial(\Phi_1,\Phi_2)}{\partial(u,v)} N_3' \right) ds\, dt.$$

Since the surface measure on $\partial\Omega'$ is given by $dS' = |\vec{N}'|\, ds\, dt$ and since

$$(N_1', N_2', N_3') = \left(-|\vec{N}'| \cos(e_1, \vec{n}'), -|\vec{N}'| \cos(e_2, \vec{n}'), -|\vec{N}'| \cos(e_3, \vec{n}') \right),$$

we get

$$\int_\Omega f\, dx\, dy\, dz = -\int_{\partial\Omega'} F \left(\frac{\partial(\Phi_1,\Phi_2)}{\partial(v,w)} \cos(e_1, \vec{n}') + \frac{\partial(\Phi_1,\Phi_2)}{\partial(w,u)} \cos(e_2, \vec{n}') + \frac{\partial(\Phi_1,\Phi_2)}{\partial(u,v)} \cos(e_3, \vec{n}') \right) dS'.$$

Evaluating the last surface integral by the divergence theorem, and using the relation

$$\frac{\partial}{\partial u}\left[F \frac{\partial(\Phi_1,\Phi_2)}{\partial(v,w)} \right] + \frac{\partial}{\partial v}\left[F \frac{\partial(\Phi_1,\Phi_2)}{\partial(w,u)} \right] + \frac{\partial}{\partial w}\left[F \frac{\partial(\Phi_1,\Phi_2)}{\partial(u,v)} \right] = f \frac{\partial(\Phi_1,\Phi_2,\Phi_3)}{\partial(u,v,w)} = f J_\Phi,$$

we finally obtain

$$\int_{\Omega} f\,dx\,dy\,dz = -\int_{\Omega'} f J_{\phi} du\,dv\,dw.$$

Since the Jacobian does not vanish in Ω', it is either positive or negative. Taking $f \equiv 1$ we see that it is negative for our choice of the sign in the normal vectors in (16). Therefore, $-J_\Phi = |J_\Phi|$. The proof for other choices of sign in (16) is similar. $\square$

Example 5.4. Consider again the spherical coordinates in $\mathbb{R}^n$:

$$\Phi : (r, \theta_1, ..., \theta_{n-1}) \mapsto (x_1, ..., x_n),$$
$$\Phi : (0, +\infty) \times (0, \pi) \times ... \times (0, \pi) \times (0, 2\pi),$$

where

$$\begin{aligned}
&x_1 = r\cos\theta_1,\\
&x_2 = r\sin\theta_1\cos\theta_2,\\
&\dots\dots\dots\dots\dots\dots\\
&x_{n-1} = r\sin\theta_1\sin\theta_2 ... \sin\theta_{n-2}\cos\theta_{n-1},\\
&x_n = r\sin\theta_1\sin\theta_2 ... \sin\theta_{n-1}.
\end{aligned}$$

Then, for a domain $D \subset \mathbb{R}^n$,

$$\int_D f(x)dx = \int_{\Phi^{-1}(D)} f \circ \Phi(r, \theta) r^{n-1} (\sin\theta_1)^{n-2} ... \sin\theta_{n-2} dr d\theta_1 ... d\theta_{n-1}.$$

If $\Gamma = RS^{n-1} = \{x \in \mathbb{R}^n : |x| = R\}$ is the sphere centred at the origin of radius R, then from (2) we have

$$\int_{RS^{n-1}} f(x)dS = R^{n-1} \int_{D'} f \circ \Phi(R, \theta)(\sin\theta_1)^{n-2} ... \sin\theta_{n-2} d\theta_1 ... d\theta_{n-1},$$

with $D' = \times(0, \pi) \times ... \times (0, \pi) \times (0, 2\pi)$. Suppose that $f \in L^1(\mathbb{R}^n)$. Then rewriting the above integral we obtain

$$\int_{\mathbb{R}^n} f(x)dx = \int_0^\infty \left(\int_{rS^{n-1}} f(x)dS \right) r^{n-1} dr.$$

This can be interpreted as integration over a sphere of radius r, and then over all concentric spheres for $0 < r < \infty$. $\diamond$

Let $A \in O(n)$ be an orthogonal matrix : $AA^t = Id$ and $\mathcal{A}$ be the corresponding linear transformation of $\mathbb{R}^n$, i.e., $\mathcal{A}(t) = At$. It follows from the formula of the change of variables in the integral that

$$\int_{S^{n-1}} f(x)dS = \int_{S^{n-1}} f \circ \mathcal{A}(t)dS = \int_{S^{n-1}} f(At)dS.$$

This property is often useful in computations of spherical integrals.