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REAL ANALYSIS LECTURE NOTES

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6. Differential Equations

6.1. **Ordinary Differential Equations.** Consider a system of d first order ordinary differential equations (ODE for short) and an initial condition

$$y' = f(t, y), \quad y(t_0) = y_0, \tag{1}$$

where $y = (y_1, \ldots, y_d)$ is a vector of unknown functions of a real variable t, $y' = (dy_1/dt, \ldots, dy_d/dt)$, and $f : \Omega \to \mathbb{R}^n$ is a continuous map on a domain $\Omega \subset \mathbb{R}^{n+1}$. We say that $y = y(t)$ defined on a t-interval J containing t_0 is a solution of the initial value problem (1) if $y(t_0) = y_0$, $(t, y(t)) \in \Omega$, $y(t)$ is differentiable and $y'(t) = f(t, y(t))$ for $t \in J$. These requirements are equivalent to the following: $y(t_0) = y_0$, $(t, y(t)) \in \Omega$, $y(t)$ is continuous and

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds, \quad t \in J.$$

Here integration should be understood component-wise.

An initial value problem involving a system of equations of m-th order

$$z^{(m)} = F(t, z, z^{(1)}, \ldots, z^{(m-1)}), \; z^{(j)}(x_0) = z_0^j \quad \text{for } j = 0, \ldots, m-1, \tag{2}$$

where $z^{(j)} = d^j z/dt^j$, z and F are n-dimensional vectors, and F is defined on an $(mn + 1)$ dimensional domain Ω, can be considered as a special case of (1), where y is a d-dimensional vector symbolically, $y = (z, z^{(1)}, \ldots, z^{(m-1)})$, or more precisely, $y = (z_1, \ldots, z_n, z_1', \ldots z_n', z_1^{(2)}, \ldots, z_n^{(m-1)})$. Correspondingly,

$$f(t, y) = (z^{(1)}, \ldots, z^{(m-1)}, F(t, y)), \quad y_0 = (z_0, \ldots, z_0^{m-1}).$$

For example, if $n = 1$, then z is a scalar, and (2) becomes

$$y_1' = y_2, \ldots, y_{m-1}' = y_m, \quad y_m' = F(t, y_1, \ldots, y_m),$$
$$y_j(t_0) = z_0^{j-1} \quad \text{for } j = 1, \ldots, m,$$

where $y_1 = z, y_2 = z', \ldots, y_m = z^{(m-1)}$.

The most fundamental question concerning ODE (1) is the *existence* and *uniqueness* of solutions.

Example 6.1. Consider the initial value problem given by $y' = y^2$ and $y(0) = c > 0$. It is easy to see that $y = \frac{c}{1-ct}$ is a solution, but it exists only on the range $-\infty < t < 1/c$, which depends on the initial condition. The initial value problem $y' = |y|^{1/2}$, $y(0) = 0$ has more than one solution, in fact, it has a one-parameter family of solutions defined by $y(t) = 0$ for $t \le c$, $y(t) = (t - c)^2/4$ for $t \ge c \ge 0$. $\diamond$

The following result gives basic conditions when a local solution of an ODE exists and is unique.

Theorem 6.1. *Let $y, f \in \mathbb{R}^d$, $f(t, y)$ be continuous on $R = \{|t-t_0| \le a, \; |y-y_0| \le b\}$ and uniformly Lipschitz continuous with respect to y. Let M be a bound for $|f(t, y)|$ on R, $\alpha = \min(a, b/M)$. Then the initial value problem* (1) *has a unique solution $y = y(t)$ on $[t_0 - \alpha, t_0 + \alpha]$.*

Proof. Let $y^0(t) \equiv y_0$. Suppose that $y^k(t)$ has been defined on $[t_0-\alpha, t_0+\alpha]$, continuous, and satisfies $|y^k(t)-y_0| \le b$ for $k=0,1,\dots,n$. Put

$$y^{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y^n(s))\, ds. \tag{3}$$

Then, since $f(t, y^n(t))$ is defined and continuous on $[t_0-\alpha, t_0+\alpha]$, the same holds for $y^{n+1}(t)$. It is clear that

$$|y^{n+1}(t) - y_0| \le \int_{t_0}^{t} |f(s, y^n(s)| ds \le M\alpha \le b.$$

Hence, $y^0(t), y^1(t), \dots$ are defined and continuous on $[t_0-\alpha, t_0+\alpha]$ and $|y^n(t)-y_0| \le b$.

It will now be verified by induction that

$$|y^{n+1}(t) - y^n(t)| \le \frac{MK^n |t-t_0|^{n+1}}{(n+1)!}, \quad \text{for} \quad t_0-\alpha \le t \le t_0+\alpha, \quad n=0,1,\dots, \tag{4}$$

where K is a Lipschitz constant for f. Clearly, (4) holds for $n=0$. Assume that it holds for up to $n-1$. Then

$$y^{n+1}(t) - y^n(t) = \int_{t_0}^{t} \left[f(s, y^n(s)) - f(s, y^{n-1}(s)\right] \, ds, \quad n \ge 1.$$

Thus, the definition of K implies that

$$|y^{n+1}(t) - y^n(t)| \le K \int_{t_0}^{t} |y^n(s) - y^{n-1}(s)|\, ds$$

and so, by (4),

$$|y^{n+1}(t) - y^n(t)| \le \frac{MK^n}{n!} \int_{t_0}^{t} |s-t_0|^n \, ds = \frac{MK^n |t-t_0|^{n+1}}{(n+1)!}.$$

This proves (4) for general n. It follows from this inequality that

$$y_0 + \sum_{n=0}^{\infty} [y_{n+1}(t) - y_n(t)] =: y(t)$$

is uniformly convergent on $[t_0-\alpha, t_0+\alpha]$, i.e., we have a uniform limit

$$y(t) = \lim_{n\to\infty} y_n(t). \tag{5}$$

Since $f(t,y)$ is uniformly continuous on R, $f(t, y_n(t)) \to f(t, y(t))$ as $n \to \infty$ on $[t_0-\alpha, t_0+\alpha]$. Thus, by taking the limit in the integral in (3) gives

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s)) ds. \tag{6}$$

Hence, (5) is a solution of (1).

In order to prove uniqueness, let $y = z(t)$ be any solution of (1) on $[t_0-\alpha, t_0+\alpha]$. Then

$$z(t) = y_0 + \int_{t_0}^{t} f(s, z(s)) ds.$$

An induction similar to that used above gives, using (3),

$$|y^n(t) - z(t)| \le \frac{MK^n |t-t_0|^{n+1}}{(n+1)!} \qquad \text{for } t_0-\alpha \le t \le t_0+\alpha, \quad n=0,1,\dots.$$

If $n \to \infty$, it follows from (5) that $|y(t)-z(t)| \le 0$, i.e., $y(t) \equiv z(t)$. This proves the theorem. $\square$

One can show that if the function f in (1) is merely continuous, then the initial value problem always has a solution, but it may not be unique. This is Peano's Existence theorem.

Consider now a *homogeneous* linear system of differential equations of the form

$$y' = A(t)y \tag{7}$$

and the corresponding *inhomogeneous* system

$$y' = A(t)y + f(t), \tag{8}$$

where $A(t)$ is a $d \times d$ matrix of continuous functions in t, and $f(t)$ is a continuous vector function of size d. It is a consequence of the theorem on existence and uniqueness of solutions of ODEs that (7) has unique solution given the initial condition $y(t_0) = y_0$. Further, if $y(t)$ is a solution of (7) and $y(t_0) = 0$ for some t_0, then $y(t) \equiv 0$. The following is immediate

Proposition 6.2 (Principle of Superposition)**.** *Let $y = y_1(t), y_2(t)$ be solutions of (7), then any linear combination $y = c_1y_1(t) + c_2y_2(t)$ with constants c_1, c_2 is also a solution. If $y = y_1(t)$ and $y = y_0(t)$ are solutions of (7) and (8) respectively, then $y = y_0(t) + y_1(t)$ is a solution of (8).*

By a *fundamental matrix* $Y(t)$ of (7) we mean a $d \times d$ matrix such that its columns are solutions of (7) and $\det Y(t) \neq 0$. If $Y = Y_0(t)$ is a fundamental matrix of solutions and C is a constant $d \times d$ matrix, then $Y(t) = Y_0(t)C$ is also a solution, in fact, any solution of (7) can be obtained this way for some suitable C.

Example 6.2. Let R be a constant $d \times d$ matrix with real coefficients. Consider the system of differential equations

$$y' = R\,y. \tag{9}$$

Let $y_1 \neq 0$ be a constant vector, and λ be a (complex) number. By substituting $y = y_1e^{\lambda t}$ into the equation we see that a necessary and sufficient condition for y to be a solution of (9) is

$$Ry_1 = \lambda y_1,$$

i.e., that λ is an eigenvalue and $y_1 \neq 0$ be a corresponding eigenvector of R. Thus to each eigenvalue λ of R there corresponds at least one solution of (9). If R has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_d$ with linearly independent eigenvectors $y_1, \ldots, y_d$, then

$$Y = \left(y_1e^{\lambda_1 t}, \ldots, y_1e^{\lambda_1 t}\right)$$

is a fundamental matrix for (9). $\diamond$

Finally, linear differential equations of higher order can be reduced to a system of first order. Indeed, let $p_j(t)$, $j = 0, \ldots, d-1$, and $h(t)$ be continuous functions. Consider the linear homogeneous equation of order d

$$u^{(d)} + p_{d-1}(t)u^{(d-1)} + \cdots + p_1(t)u' + p_0(t)u = 0 \tag{10}$$

and the corresponding inhomogeneous equation

$$u^{(d)} + p_{d-1}(t)u^{(d-1)} + \cdots + p_1(t)u' + p_0(t)u = h(t). \tag{11}$$

We let $y = (u, u^{(1)}, \ldots, u^{(d-1)})$,

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & \cdots & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -p_0 & -p_1 & -p_2 & -p_3 & \cdots & -p_{d-1} \end{pmatrix},$$

and $f(t) = (0, \dots, 0, h(t))$. With this choice of u, A and f, equation (10) becomes equivalent to (7), while (11) equivalent to (8). With this we can apply results available for first order systems, in particular, given the initial conditions $u(t_0) = u_0$, $u'(t_0) = u'_0$, ..., $u^{(d-1)}(t_0) = u_0^{(d-1)}$, where u_0^j are arbitrary numbers, the corresponding initial value problem has a unique solution. The Principle of Superposition also holds: let $u = u_1(t), u_2(t)$ be two solutions of (10), then any linear combination $u(t) = c_1u_1(t) + c_2u_2(t)$ is also a solution, and $u(t) = u_1(t) + u_0(t)$ is a solution of (11) if $u_0(t)$ is.

6.2. **Partial Differential Equations.** A partial differential equation (PDE for short) is an equation that involves an unknown function of two or more independent variables and certain partial derivatives of the unknown function. More precisely, let u denote a function of n independent variables $x_1, \dots, x_n$, $n \geq 2$. Then a relation of the form

$$F\left(x_1, \dots, x_n, u, u_{x_1}, \dots, u_{x_n}, u_{x_1x_2}, \dots\right) = 0, \tag{12}$$

where F is a function of its arguments, is a partial differential equation in u. The following equations are some examples of PDEs on $\mathbb{R}^2$ with coordinates (x, y):

$$xu_x + yu_y - 2u = 0 \tag{13}$$

$$yu_x - xu_y = x \tag{14}$$

$$u_{xx} - u_y - u = 0 \tag{15}$$

$$uu_x + yu_y - u = xy^2 \tag{16}$$

$$u_{xx} + x(u_y)^2 + yu = y \tag{17}$$

A typical problem in the theory of PDEs is for a given equation to find on some domain of $\mathbb{R}^n$ a solution satisfying certain additional initial or boundary conditions. Analogous to ODEs, the highest-order derivative appearing in a PDE is called the order of the equation. Thus, (13), (14), and (16) are all first-order PDEs, and the remaining two are second-order. If there exists a function u defined in a domain under consideration, such that u and its derivatives identically satisfy (12), then u is called a solution of the equation.

A PDE is called linear if it is at most of the first order in u and its derivatives. Equation (13), (14), and (15) above are linear, while the other two are not.

Example 6.3. The first-order linear ODE of the form

$$\frac{du}{dx} + u = f(x)$$

has the general solution

$$u(x) = \int_0^x e^{-(x-t)} f(t)dt + Ce^{-x}.$$

Now if we consider the first order PDE on $\mathbb{R}^2$ for the unknown function $u = u(x, y)$,

$$\frac{\partial u}{\partial x} + u = f(x), \tag{18}$$

then its general solution is given by

$$u(x) = \int_0^x e^{-(x-t)} f(t)dt + g(y)e^{-x},$$

where $g(x)$ is an arbitrary function of y. It is easy to see that for any choice of g, the function u satisfies (18). Thus the general solution of a PDE may contain some arbitrary functions, not necessarily constants. $\diamond$

A map between function spaces that involves differentiation is called a *differential operator*. For example, the map $L : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$ given by $L(u) = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$ is a second-order differential operator. A differential operator L is called linear if $L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2)$. It is immediate that L is linear if and only if the PDE $L(u) = 0$ is linear, and that any finite sum of linear differential operators is again linear. Given a linear operator L, the PDE $Lu = 0$ is called a homogeneous PDE, while $Lu = f$ is inhomogeneous for $f \not\equiv 0$. As for ODEs, the Principle of Superposition also holds: a linear combination of solutions of a homogeneous equation is again a solution, and the sum of solutions of a homogeneous and inhomogeneous equation is a solution of the inhomogeneous one.

We now consider three classical PDEs: the Wave equation, the Heat equation, and the Laplace equation. For a function $u = u(x, t)$, where $t \in \mathbb{R}$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, the equation of the form

$$\frac{\partial^2 u}{\partial t^2} - c^2 \left(\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} \right) = \frac{\partial^2 u}{\partial t^2} - c^2 \Delta_x u = 0 \quad c = const., \tag{19}$$

is called the Wave equation. This equation describes many types of elastic and electromagnetic waves. In many physical application the variable t represents time and x represents coordinates in the Euclidean space where the physical experiment takes place. As an equation of second order, a typical initial condition for the Wave equation is of the form

$$\begin{aligned} u(x, 0) &= f(x), \\ u_t(x, 0) &= g(x). \end{aligned}$$

When $n = 1$ the equation has surprisingly simple solution. We let $\xi = x - ct$ and $\eta = x + ct$. After this change of variable equation (19) has the form $u_{\xi\eta} = 0$, which after elementary considerations admits the solution $u(\xi, \eta) = F(\xi) + G(\eta)$, or after returning to the origin variables,

$$u(x, t) = F(x - ct) + G(x + ct).$$

The general solution for an arbitrary n can be obtained using the theory of Fourier series.

Consider now the equation of the form

$$\frac{\partial u}{\partial t} - k \Delta_x u = 0. \tag{20}$$

This is the so-called Heat equation. Again $t \in \mathbb{R}$ is the "time" variable, $x = (x_1, \ldots x_n)$, and Δ_x is the Laplacian in variable x. As a primary physical application, equation (20) describes the conduction of heat with the function u usually representing the temperature of a "point" with coordinates x at time t, but more generally it governs a range of physical phenomena described as *diffusive*. Typical initial conditions are

$$\begin{aligned} u(x, 0) &= f(x), \\ u(0, t) &= 0, \end{aligned}$$

where the first condition can be interpreted as the initial temperature of the system, and the second one declares that the temperature is fixed at the "end" point. Solution of (20) for $n = 1$ can be found, for example, using the *separation of variables* method: assuming $k = 1$ we seek solution of the form $u(x, t) = X(x)T(t)$. Putting this into (20) gives

$$\frac{T'(t)}{T(t)} = \frac{X'(x)}{X(x)}.$$

Since the right-hand side is independent of t and the left-hand side of x, each side must be a constant. This gives

$$T' = \lambda T, \quad X'' = \lambda X.$$

Solving these equations gives

$$u(x,t) = e^{\lambda t}(Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}).$$

Initial conditions will then specify the values of the constants.

Finally, for a function $u = u(x_1, \dots, x_n)$ consider the equation

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0. \tag{21}$$

This PDE, which is called the Laplace equation, has many applications in gravitation, elastic membranes, electrostatics, fluid flow, etc, as well as numerous applications in other areas of pure mathematics. There are two types of boundary conditions on a bounded domain $\Omega \subset \mathbb{R}^n$ for the Laplace equation that give a well-posed problem: the Dirichlet condition

$$u(x) = f(x), \quad x \in \partial\Omega,$$

and the Neumann condition

$$\frac{\partial u}{\partial \vec{n}}(x) = f(x), \quad x \in \partial\Omega,$$

where $\frac{\partial u}{\partial \vec{n}}(x)$ is the derivative in the normal direction to $\partial\Omega$. Solutions of (21) are called *harmonic* functions. For $n = 2$, solutions can be found again by separation of variables.

Unlike the theory of ODE, not every PDE has a solution. In 1957 H. Lewy found an example of a first order linear PDE that has no solution. The corresponding (complex-valued) differential operator on $C^\infty(\mathbb{R}^3)$ is

$$Lu = -u_x - iu_y + 2i(x+iy)u_z.$$

Then there exists a real valued function $f(x,y,z)$ of class $C^\infty(\mathbb{R}^3)$ such that the equation $Lu = f(x,y,z)$ has no solution of class $C^1(\Omega)$ in any open subset $\Omega \subset \mathbb{R}^3$. While Lewy's example is not explicit, later explicit constructions were also found.

The situation is different, however, if the functions involved in a PDE are real-analytic. Recall that for a domain $\Omega \subset \mathbb{R}^n$ a function $f(x) : \Omega \to \mathbb{R}$ is real-analytic if near any point in Ω it can be represented by a convergent power series. More precisely, if $a = (a_1, \dots, a_n) \in \Omega$, there exists a polydisc $U(a,r) = \{x \in \mathbb{R}^n : |x_j - a_j| < r,\ j = 1, \dots, n\}$, $U \subset \Omega$ such that

$$f(x) = \sum_{|k|=0}^{\infty} b_k (x-a)^k,$$

where $k = (k_1, k_2, \dots, k_n)$ is the multi-index, $|k| = k_1 + k_2 + \cdots + k_n$, $a_k \in \mathbb{R}$, and

$$(x-a)^k = (x_1 - a_1)^{k_1} \cdot \ldots \cdot (x_n - a_n)^{k_n}.$$

The coefficients a_k of the power series are, in fact, the Taylor coefficients of f that can be computed by the formula

$$b_k = \frac{1}{k!} D^k f(a) = \frac{1}{k_1! k_2! \dots k_n!} \, \frac{\partial^{|k|} f}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}}(a).$$

The space of real-analytic functions is denoted by $C^\omega(\Omega)$. Every real-analytic function in infinitely differentiable, but the converse is not true, for example the function $h(x)$ on $\mathbb{R}$ given by

$$h(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ e^{-1/x}, & \text{if } x > 0 \end{cases}$$

is of class $C^\infty(\mathbb{R})$ as repeated application of the L'Hôspital rule shows, but it is not real analytic at the origin. The reason is that all derivatives of h at zero vanish, but h is not identically zero on any neighbourhood of the origin, and so h cannot be represented by its Taylor series. A map $f : \mathbb{R}^n \to \mathbb{R}^m$ is real-analytic if every component of f is a real-analytic function.

Theorem 6.3 (Cauchy-Kovalevskaya theorem)**.** *Consider the system of partial differential equations*

$$\frac{\partial u_i}{\partial x_n} = \sum_{k=1}^{n-1} \sum_{j=1}^{N} a_{ij}^k(x_1, \ldots, x_{n-1}, u_1, \ldots, u_N) \frac{\partial u_j}{\partial x_k} + b_i(x_1, \ldots, x_{n-1}, u_1, \ldots, u_N), \quad i = 1, \ldots, N \tag{22}$$

with the initial condition

$$u_i = 0 \quad \textit{on} \quad x_n = 0, \quad i = 1, \ldots, N. \tag{23}$$

Let the functions a_{ij}^k and b_i be real analytic at the origin of $\mathbb{R}^{N+n-1}$. Then the system (22) *with the initial condition* (23) *has a unique (among real-analytic functions) system of solutions u_i that is real analytic at the origin.*

The proof of the Cauchy-Kovalevskaya theorem can be found in comprehensive textbooks on PDEs.