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REAL ANALYSIS LECTURE NOTES

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7. The space of test functions

7.1. **Space of test functions.** Let Ω be an open subset of $\mathbb{R}^n$. For an integer $k \geq 0$ we denote by $C^k(\Omega)$ the linear space of functions $h : \Omega \longrightarrow \mathbb{R}$ that have continuous partial derivatives up to order k on Ω. We define $C^\infty(\Omega) = \cap_{k=0}^\infty C^k(\Omega)$ to be the space of real functions admitting continuous partial derivatives of any order. The standard topology on $C^\infty(\Omega)$ can be defined by the following notion of convergence: we say that a sequence $\{h^k\} \subset C^\infty(\Omega)$ converges as $k \longrightarrow \infty$ to a function h in $C^\infty(\Omega)$ if h^k converges to h uniformly along with all partial derivatives of any order on any compact subset of Ω. If K is a compact subset of Ω, we use the standard norm

$$\| u \|_{C^k(K)} = \sum_{|\alpha| \leq k} \sup_{x \in K} |D^\alpha u(x)|$$

for $u \in C^k(\Omega)$.

Recall that $\operatorname{supp} h$, the *support* of a function $h \in C^0(\mathbb{R}^n)$, is defined to be the closure in $\mathbb{R}^n$ of the set $\{x \in \Omega : h(x) \neq 0\}$. Denote by $C_0^\infty(\Omega)$ the subspace of $C^\infty(\mathbb{R}^n)$ which consists of functions h such that $\operatorname{supp} h$ is a compact subset of Ω.

Our main goal is to introduce and to study the space of continuous linear functionals on the linear vector space $C_0^\infty(\Omega)$. For this we first need to choose some topology on $C_0^\infty(\Omega)$. For our purposes it will be sufficient simply to define the notion of convergence of a sequence of elements in $C_0^\infty(\Omega)$. This will allow us to define the continuity of linear functionals. We say that *a sequence of functions* (φ_j) *of class* $C_0^\infty(\Omega)$ *converges to* $\varphi \in C_0^\infty(\Omega)$ if the following conditions hold:

- (i) There exists a compact subset K such that $K \subset \Omega$ and $\varphi_j = 0$ on $\mathbb{R}^n \backslash K$ for every $j = 1, 2, 3....$ In other words, $\operatorname{supp} \varphi_j \subset K \subset \Omega$ for every j.
- (ii) For every α the sequence $D^\alpha \varphi_j$ converges to $D^\alpha \varphi$ uniformly on K. That is all partial derivatives of φ_j of all orders converge uniformly to the corresponding partial derivatives of φ.

Definition 7.1. *The space $C_0^\infty(\Omega)$ equipped with the above topology of convergence of sequences will be denoted by $\mathcal{D}(\Omega)$. The elements of this space are called test-functions.*

We leave as an exercise for the reader to verify that $\mathcal{D}(\Omega)$ is a topological vector space.

Example 7.1. Denote by $|x| = \sqrt{|x_1|^2 + ... + |x_n|^2}$ the Euclidean norm on $\mathbb{R}^n$. Set

$$\omega_\varepsilon(x) = \begin{cases} C_\varepsilon \, e^{-\frac{\varepsilon^2}{\varepsilon^2 - |x|^2}}, & |x| < \varepsilon, \\ 0, & |x| \geq \varepsilon. \end{cases}$$

Here the constant C_ε is determined by the condition

$$\int_{\mathbb{R}^n} \omega_\varepsilon(x) dx = 1,$$

that is,

$$C_\varepsilon = \varepsilon^{-n}\left(\int_{B(0,1)} e^{\frac{-1}{1-|t|^2}}\,dt\right)^{-1},$$

where $B(0,1)$ denotes the Euclidean unit ball in $\mathbb{R}^n$. This is a model example of a function from $\mathcal{D}(\mathbb{R}^n)$ (the so-called *bump function*). Observe that

$$\omega_\varepsilon(x) = \varepsilon^{-n}\omega_1(x/\varepsilon).$$

◇

In what follows we write $\omega(x)$ instead of $\omega_1(x)$. The bump function allows us to construct suitable test-functions for an arbitrary domain Ω in $\mathbb{R}^n$. We denote by $\Omega^\varepsilon := \cup_{x\in\Omega} B(x,\varepsilon)$, the ε-neighbourhood of Ω.

Lemma 7.2. *Given a compact subset $\Omega \subset \mathbb{R}^n$ and $\varepsilon > 0$ there exists a function $\eta : \mathbb{R}^n \longrightarrow [0,1]$ of class $C^\infty(\mathbb{R}^n)$ such that*

$$\eta(x) = 1, \quad \text{for } x \in \Omega^\varepsilon,$$
$$\eta(x) = 0, \quad \text{for } x \in \mathbb{R}^n\backslash\Omega^{3\varepsilon}.$$

Proof. Let

$$\lambda(x) = \begin{cases} 1, & x \in \Omega^{2\varepsilon}, \\ 0, & x \in \mathbb{R}^n\backslash\Omega^{2\varepsilon}, \end{cases}$$

be the characteristic function of $\Omega^{2\varepsilon}$. Then the function defined by the convolution integral

$$\eta(x) = \int_{\mathbb{R}^n} \lambda(y)\omega_\varepsilon(x-y)dy$$

satisfies the required conditions. □

Corollary 7.3. *Let Ω be a bounded domain and Ω' be its subset such that $\overline{\Omega'} \subset \Omega$. Then there exists a function $\eta \in \mathcal{D}(\Omega)$, $\eta : \Omega \longrightarrow [0,1]$ such that $\eta(x) = 1$ whenever $x \in \Omega'$.*

We point out that the introduced above topology on the space $\mathcal{D}(\Omega)$ is not metrizable. In other words, one cannot define a distance d on the space of test functions such that the convergence with respect to d is equivalent to the introduced above convergence of a sequence of test functions. To see this, recall the following elementary assertion.

Claim. *Let $\{u_k^m\}$ be a countable family of sequences in a metric space (X,d). Suppose that for every $m = 0,1,...$ the sequence $\{u_k^m\}$ admits the limit $u^m := \lim_{k\to\infty} u_k^m$ and the sequence of these limits $\{u^m\}$ also tends to $u \in X$ as $m \to \infty$. Then for every m there exists k_m such that $k_{m-1} < k_m$ and the sequence $\{u_{k_m}^m\}$ converges to u.*

We leave an easy proof as an exercise. Consider now in $\mathcal{D}(\mathbb{R})$ the functions

$$u_k^m(x) = \begin{cases} \frac{1}{k}e^{-\frac{m^2}{m^2-x^2}}, & \text{for } |x| \le m, \\ 0, & \text{for } |x| \ge m. \end{cases}$$

For every m we have $\lim_{k\to\infty} u_k^m = 0$ in $\mathcal{D}(\mathbb{R})$, but for any choice of k_m the sequence $\{u_{k_m}^m\}$ does not converge in $\mathcal{D}(\mathbb{R})$ since the supports of these test functions are not uniformly bounded. This shows that the topology of $\mathcal{D}(\mathbb{R}^n)$ is not metrizable.

It is not difficult to describe the topology on $\mathcal{D}(\mathbb{R}^n)$ defined by the introduced above convergence. Consider for simplicity the case $n = 1$, as the general case is similar. Given m consider $m+1$ positive continuous functions γ_j. Define a neighbourhood of the origin in $\mathcal{D}(\mathbb{R})$ as the set of test-functions ϕ satisfying $|\phi^{(j)}(x)| \leq \gamma_j(x)$, $x \in \mathbb{R}$, $j = 1, 2, \ldots, m$. Then the convergence in this topology precisely coincides with the introduced above convergence on $\mathcal{D}(\mathbb{R})$. The reader is encouraged to supply a proof of this fact.

7.2. **Regularization of functions.** The convolution $f*g$ of two functions $f, g \in L^2(\mathbb{R}^n)$ is defined by the integral

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy = \int_{\mathbb{R}^n} f(x-y)g(y)dy. \tag{1}$$

The convolution has particularly nice properties when the second factor is a test function. Denote by $L^1_{loc}(\Omega)$ the space of locally Lebesgue-integrable functions on Ω. Recall that a measurable function f is in $L^1_{loc}(\Omega)$ if and only if $\int_X |f(x)|dx < \infty$ for every compact measurable subset $X \subset \Omega$. Let $f \in L^1_{loc}(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Then for every $x \in \mathbb{R}^n$ the function $y \mapsto \varphi(x-y)$ is a test-function and so the convolution

$$f * \varphi(x) = \int_{\mathbb{R}^n} f(y)\varphi(x-y)dy \tag{2}$$

is well-defined point-wise as a usual function on all of $\mathbb{R}^n$.

Proposition 7.4. *We have* $f * \varphi \in C^\infty(\mathbb{R}^n)$ *and*

$$D^\alpha(f * \phi) = f * D^\alpha \varphi. \tag{3}$$

Proof. (a) Let us show that the convolution $f * \varphi$ is a continuous function. Let $x^k \in \mathbb{R}^n$ be a sequence converging to x. Then $\varphi(x^k - y) \longrightarrow \varphi(x-y)$, as $k \to \infty$ everywhere, and

$$f * \varphi(x^k) = \int f(y)\varphi(x^k - y)dy \longrightarrow \int f(y)\varphi(x-y)dy = (f*\varphi)(x), \text{ as } k \to \infty,$$

by the Lebesgue Convergence theorem.

(b) Denote by $\vec{e}_j$, $j = 1, ..., n$, the vectors of the standard basis of $\mathbb{R}^n$. For a fixed $x \in \mathbb{R}^n$ we have

$$\frac{1}{t}(\varphi(x + t\vec{e}_j - y) - \varphi(x-y)) \longrightarrow \frac{\partial}{\partial x_j}(\varphi(x-y)), \; t \to 0,$$

everywhere (this follows from the Mean Value theorem applied to the function $\varphi(x + t\vec{e}_j - y)$ on $[0, t]$). Therefore, by the Lebesgue Convergence theorem,

$$\frac{1}{t}(f * \varphi(x + t\vec{e}_j) - f * \varphi(x)) = \int f(y)\frac{1}{t}(\varphi(x + t\vec{e}_j - y) - \varphi(x-y))dy \longrightarrow \int f(y)\left(\frac{\partial}{\partial x_j}\varphi\right)(x-y)dy$$

Hence, the partial derivative of $f * \varphi$ with respect to x_j exists and

$$\frac{\partial}{\partial x_j} f * \varphi = \int f(y)\left(\frac{\partial \varphi}{\partial x_j}\right)(x-y)dy = f * \left(\frac{\partial \varphi}{\partial x_j}\right).$$

Since $\frac{\partial}{\partial x_j}\varphi \in \mathcal{D}(\mathbb{R}^n)$, it follows from part (a) that the partial derivative $\frac{\partial}{\partial x_j}(f * \varphi)$ is continuous. Proceeding by induction, we obtain that $f * \varphi \in C^\infty(\mathbb{R}^n)$ and satisfies (3). $\square$

A special case, important in applications, arises if we take the bump-function ω_ε as φ in (2).

Definition 7.5. *The convolution* $f_\varepsilon := f * \omega_\varepsilon$ *is called the regularization of a function* $f \in L^1_{loc}(\mathbb{R}^n)$.

Proposition 7.6. *We have*

(i) $f_\varepsilon \in C^\infty(\mathbb{R}^n)$;

(ii) *if* $f \in C(\mathbb{R}^n)$ *then* $f_\varepsilon \longrightarrow f$, *as* $\varepsilon \to 0+$, *in* $C(\overline{\Omega})$ *for every bounded subset* Ω *of* $\mathbb{R}^n$*;*
(iii) *if* $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, *then* $f_\varepsilon \longrightarrow f$, *as* $\varepsilon \longrightarrow 0+$, *in* $L^p(\mathbb{R}^n)$.

Proof. Part (i) follows from Proposition 7.4. To show (ii), assume that f is a continuous function on $\mathbb{R}^n$. Then

$$f_\varepsilon(x) = \int_{\mathbb{R}^n} \omega_\varepsilon(x-y)f(y)dy = \int_{\mathbb{R}^n} \omega_\varepsilon(y)f(x-y)dy = \int_{|y|\leq\varepsilon} \omega_\varepsilon(y)f(x-y)dy. \tag{4}$$

Since

$$\int_{\mathbb{R}^n} \omega_\varepsilon(x)dx = 1,$$

we have

$$\left|\int_{\mathbb{R}^n} \omega_\varepsilon(y)f(x-y)dy - f(x)\right| = \left|\int_{\mathbb{R}^n} \omega_\varepsilon(y)(f(x-y)-f(x))dy\right| \leq M_\varepsilon \int_{|y|\leq\varepsilon} \omega_\varepsilon(y)dy = M_\varepsilon,$$

where

$$M_\varepsilon = \sup_{x\in\Omega, |y|\leq\varepsilon} |f(x-y)-f(x)|.$$

Since f is continuous on the compact subset $\overline{\Omega^\varepsilon}$ of $\mathbb{R}^n$, it is uniformly continuous there so that $M_\varepsilon \longrightarrow 0$ as $\varepsilon \to 0+$. This proves (ii).

For the proof of (iii) we need to show that $\| f_\varepsilon - f \|_{L^p} \longrightarrow 0$, as $\varepsilon \longrightarrow 0+$, where

$$\| h \|_{L^p} = \| h \|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |h(x)|^p dx\right)^{1/p}.$$

Lemma 7.7. *For every* $\varepsilon > 0$ *we have*

$$\| f_\varepsilon \|_{L^p} \leq \| f \|_{L^p}.$$

Proof. Consider first the case $p = 1$. By Fubini's theorem

$$\| f_\varepsilon \|_{L^1} = \int |f * \omega_\varepsilon(x)|dx \leq \int\int |f(y)|\omega_\varepsilon(x-y)dydx = \int |f(y)| \left(\int \omega_\varepsilon(x-y)dx\right) dy$$
$$= \int |f(y)|dy = \| f \|_{L^1}.$$

Let now $p > 1$. For p and q satisfying $\frac{1}{p} + \frac{1}{q} = 1$ we have, by Hölder's inequality,

$$|f * \omega_\varepsilon|^p \leq \left(\int |f(y)|\omega_\varepsilon(x-y)dy\right)^p = \left(\int |f(y)|\omega_\varepsilon(x-y)^{1/p}\omega_\varepsilon(x-y)^{1/q}dy\right)^p$$
$$\leq \left(\int |f(y)|^p\omega_\varepsilon(x-y)dy\right)\left(\int \omega_\varepsilon(x-y)dy\right)^{p/q} = \int |f(y)|^p\omega_\varepsilon(x-y)dy.$$

Therefore, by Fubini's theorem,

$$\left(\int |f * \omega_\varepsilon(x)|^p dx\right)^{1/p} \leq \left(\int\left(\int |f(y)|^p\omega_\varepsilon(x-y)dy\right)dx\right)^{1/p}$$
$$= \left(\int |f(y)|^p\left(\int \omega_\varepsilon(x-y)dx\right)dy\right)^{1/p} = \left(\int |f(y)|^p dy\right)^{1/p},$$

which proves the lemma. □

Now let $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$. From the theory of Lebesgue integral, given $\tau > 0$ there exists $A > 0$ and a function $h \in C(\mathbb{R}^n)$ with $\operatorname{supp} h \subset \{|x| \le A\}$ such that

$$\| f - h \|_{L^p} \le \tau.$$

Then by Lemma 7.7 we have

$$\| f_\varepsilon - h_\varepsilon \|_{L^p} = \| (f-h)_\varepsilon \|_{L^p} \le \| f - h \|_{L^p} \le \tau$$

for every ε. Furthermore, $\operatorname{supp} h_\varepsilon \subset K = \{|x| \le A+1\}$ for ε small enough. Hence, by (ii), $h_\varepsilon \longrightarrow h$ as $\varepsilon \longrightarrow 0+$ in $C(K)$. It follows that

$$\| h - h_\varepsilon \|_{L^p} \le \tau$$

for all ε sufficiently small. Thus,

$$\| f - f_\varepsilon \|_{L^p} \le \| f - h \|_{L^p} + \| h - h_\varepsilon \|_{L^p} + \| h_\varepsilon - f_\varepsilon \|_{L^p} \le 3\tau,$$

which concludes the proof of the proposition. □

7.3. **Partition of unity.** A family of open subsets $(U_\alpha)_{\alpha \in A}$ in $\mathbb{R}^n$ is called a *covering* of a subset $X \subset \mathbb{R}^n$ if $X \subset \cup_{\alpha \in A} U_\alpha$. A covering is called *locally finite* if every point $x \in X$ admits a neighbourhood V such that the intersection $V \cap U_\alpha$ is not empty only for a finite subset of α in A. A covering $(V_\beta)_{\beta \in B}$ is called a *subcovering* of $(U_\alpha)_{\alpha \in A}$ if for every $\beta \in B$ there exists $\alpha \in A$ such that $V_\beta \subset U_\alpha$. We denote by X° the interior of X.

Proposition 7.8. *Let X be an open subset of $\mathbb{R}^n$ and (U_α) be its covering. Then there exists a locally finite subcovering (V_β) of (U_α) with the following property: for every β the set V_β contains the closure of some ball $B(p_\beta, r_\beta)$ and these open balls $(B(p_\beta, r_\beta)$ form a locally finite subcovering $(W_\gamma)_{\gamma \in \Gamma}$ of (V_β).*

Proof. We set

$$K_j = \{z \in X : |x| \le j, \operatorname{dist}(x, \partial X) \ge 1/j\}.$$

Then K_j is a sequence of compact subsets of X satisfying

(i) $K_j \subset K_{j+1}^\circ$
(ii) $X = \cup_{j=1}^\infty K_j$.

Since $K_i \backslash K_{i-1}^\circ$ is compact, one can choose $U_{\alpha_1}^i, ..., U_{\alpha_{k_i}}^i$ a finite subset of (U_α) with

$$(K_i \backslash K_{i-1}^\circ) \subset \cup_{j=1}^{k_i} U_{\alpha_j}^i.$$

Set $V_j^i = U_{\alpha_j}^i \cap (K_i \backslash K_{i-2})^\circ$. Then (V_j^i) is a locally finite subcovering. For every $p \in V_\beta$ consider a ball $B(p, r(p))$ contained in V_β for all β with $p \in V_\beta$. Then this family of balls form a subcovering of (V_β) and we apply a previous construction extracting a locally finite subcovering by balls. □

Corollary 7.9. *Let X be an open subset of $\mathbb{R}^n$ and (U_α) be its covering. Then there exists a locally finite subcovering (W_γ) and a family of functions $\eta_\gamma \in C_0^\infty(W_\gamma)$ with the following properties:*

(i) *$\eta_\gamma(x) \ge 0$ for every $x \in X$ and every γ.*
(ii) *$\sum_\gamma \eta_\gamma(x) = 1$ for every $x \in X$.*

Proof. Consider locally finite subcoverings (W_γ) and (V_β) constructed in the previous proposition. Fix γ. By construction, there exists a finite number of β such that the closed ball $\overline{W_\gamma}$ is contained in V_β. By Corollary 7.3 there exists a function

$$\phi_\gamma \in C_0^\infty\Big(\bigcap_{W_\gamma \subset V_\beta} V_\beta\Big)$$

with values in $[0,1]$ that is equal to 1 on W_γ. Hence, $\phi(x) = \sum_\gamma \phi_\gamma(x)$ is well-defined for every $x \in X$ and $\phi(x) > 0$. Now set $\eta_\gamma(x) = \phi_\gamma(x)/\phi(x)$. $\square$

The family (η_γ) is called *a partition of unity* subordinated to the covering (W_γ).