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REAL ANALYSIS LECTURE NOTES

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9. Differentiation of distributions and the structure theorems

We saw in the previous section that the space of distributions is a generalization of the space of functions defined point-wise. A remarkable consequence of this fact is that all distributions admit partial derivatives of any order (suitably defined).

9.1. **Definition, basic properties, first examples.** We begin with some motivation. Suppose that f is a regular function on a domain Ω in $\mathbb{R}^n$, say, of class $C^1(\Omega)$. Then its partial derivative (in the usual sense) $\frac{\partial f}{\partial x_j}$ defines a distribution acting on $\varphi \in \mathcal{D}(\Omega)$ by

$$\langle T_{\frac{\partial f}{\partial x_j}}, \varphi \rangle = \int_\Omega \frac{\partial f(x)}{\partial x_j} \varphi(x) dx = -\int_\Omega f(x) \frac{\partial \varphi(x)}{\partial x_j} dx = -\langle T_f, \frac{\partial \varphi}{\partial x_j} \rangle,$$

where the second equality follows from the integration by parts formula. But the last expression is defined for *an arbitrary* distribution f; so it is natural to take it as a definition of the derivative of a distribution. For $f \in \mathcal{D}'(\Omega)$ and a multi-index $\alpha = (\alpha_1, ..., \alpha_n)$ we set

$$\langle D^\alpha f, \varphi \rangle := (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle,$$

where we used the usual notation

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1} ... \partial x^{\alpha_n}}, \quad |\alpha| = \alpha_1 + ... + \alpha_n.$$

Derivatives in $\mathcal{D}'(\Omega)$ are often called *weak* derivatives. It is easy to check (do it!) that weak differentiation is a well-defined operation, that is, $D^\alpha f \in \mathcal{D}'(\Omega)$. We note some basic properties of this operation:

(0) If $f \in C^1(\Omega)$, then $\frac{\partial}{\partial x_j} T_f = T_{\frac{\partial f}{\partial x_j}}$.

(1) The map $D^\alpha : \mathcal{D}'(\Omega) \longrightarrow \mathcal{D}'(\Omega)$ is linear and continuous. The linearity is obvious. In order to prove continuity, consider a sequence $f_j \longrightarrow 0$ in $\mathcal{D}'(\Omega)$ as $j \to \infty$. Then for any $\varphi \in \mathcal{D}(\Omega)$,

$$\langle D^\alpha f_j, \varphi \rangle = (-1)^{|\alpha|} \langle f_j, D^\alpha \varphi \rangle \longrightarrow 0, \quad \text{as } j \to \infty.$$

Thus, if a sequence (f_j) converges to f in $\mathcal{D}'(\Omega)$, then all partial derivatives of f_j converge to the corresponding partial derivatives of f.

(2) Every distribution admits partial derivatives of all orders.

(3) For any multi-indices α and β we have

$$D^{\alpha+\beta} f = D^\alpha (D^\beta f) = D^\beta (D^\alpha f).$$

(4) The Leinbitz rule. If $f \in \mathcal{D}'(\Omega)$ and $a \in C^\infty(\Omega)$ then

$$\frac{\partial (af)}{\partial x_k} = a \frac{\partial f}{\partial x_k} + \frac{\partial a}{\partial x_k} f.$$

Indeed, given $\varphi \in \mathcal{D}(\Omega)$ we have

$$\langle\frac{\partial(af)}{\partial x_k},\varphi\rangle = -\langle af,\frac{\partial\varphi}{\partial x_k}\rangle = -\langle f,a\frac{\partial\varphi}{\partial x_k}\rangle = -\langle f,\frac{\partial(a\varphi)}{\partial x_k} - \frac{\partial a}{\partial x_k}\varphi\rangle =$$
$$-\langle f,\frac{\partial(a\varphi)}{\partial x_k}\rangle + \langle f,\frac{\partial a}{\partial x_k}\varphi\rangle = \langle\frac{\partial f}{\partial x_k},a\varphi\rangle + \langle\frac{\partial a}{\partial x_k}f,\varphi\rangle$$
$$= \langle a\frac{\partial f}{\partial x_k},\varphi\rangle + \langle\frac{\partial a}{\partial x_k}f,\varphi\rangle = \langle\left(a\frac{\partial f}{\partial x_k} + \frac{\partial a}{\partial x_k}f\right),\varphi\rangle.$$

We consider several elementary examples in dimension 1.

Example 9.1. Consider the so-called *Heaviside* function

$$\theta(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x \le 0. \end{cases}$$

Then,

$$\langle\theta',\phi\rangle = -\langle\theta,\phi'\rangle = -\int_0^\infty \phi'(x)dx = \phi(0) = \langle\delta,\phi\rangle.$$

Thus, $\theta' = \delta$. $\diamond$

Example 9.2. More generally, let f be a function of class C^1 on $(-\infty, x_0]$ and of class C^1 on $[x_0,\infty)$. Denote by $[f]_{x_0} := f(x_0+0) - f(x_0-0)$ its "jump" at x_0. Denote also by $T_{f'}$ the regular distribution defined by the usual derivative f' of f. We claim that

$$f' = T_{f'} + [f]_{x_0}\delta(x-x_0),$$

where the derivative f' of f on the left is understood in the sense of distributions. For any $\varphi \in \mathcal{D}'(\mathbb{R})$ we have

$$\langle f',\varphi\rangle = -\langle f,\varphi'\rangle = -\int f(x)\varphi'(x)dx = [f]_{x_0}\varphi(x_0) + \int f'(x)\varphi(x)dx$$
$$= \langle[f]_{x_0}\delta(x-x_0) + T_{f'},\varphi\rangle.$$

$\diamond$

Example 9.3. Let $f(x) = \ln|x|$. Then for every $\phi \in \mathcal{D}(\mathbb{R})$ we obtain

$$\langle\ln|x|',\varphi\rangle = -\langle\ln|x|,\varphi'\rangle = -\int_{\mathbb{R}}\ln|x|\varphi' dx =$$
$$-\lim_{\varepsilon\longrightarrow 0+}\left(\int_{-\infty}^{-\varepsilon}\ln|x|\varphi'(x)dx + \int_{\varepsilon}^{+\infty}\ln|x|\varphi'(x)dx\right) =$$
$$-\lim_{\varepsilon\longrightarrow 0+}\left(\ln\varepsilon\varphi(-\varepsilon) - \int_{-\infty}^{-\varepsilon}\frac{\varphi(x)}{x}dx - \ln\varepsilon\varphi(\varepsilon) - \int_{\varepsilon}^{\infty}\frac{\varphi(x)}{x}dx\right) =$$
$$-\lim_{\varepsilon\longrightarrow 0+}\left(\ln\varepsilon[\varphi(-\varepsilon) - \varphi(\varepsilon)] - \int_{|x|\ge\varepsilon}\frac{\varphi(x)}{x}dx\right) =$$
$$\lim_{\varepsilon\longrightarrow 0+}\int_{|x|\ge\varepsilon}\frac{\varphi(x)}{x}dx = \langle\mathcal{P}\frac{1}{x},\varphi\rangle$$

Thus

$$\ln|x|' = \mathcal{P}\frac{1}{x}.$$

$\diamond$

Example 9.4. We have

$$\langle \delta', \phi \rangle = -\langle \delta, \phi' \rangle = -\phi'(0).$$

$\diamond$

9.2. **Basic differential equations with distributions.** We saw in the previous examples that the usual point-wise derivative does not give a full information about the derivative in the sense of distributions: the Dirac delta-function appears at the points of discontinuity. The following important statement shows that this does not happen for derivatives in the sense of distributions.

Theorem 9.1. *Let $f \in \mathcal{D}'((a,b))$ and $f' = 0$ in $\mathcal{D}'((a,b))$. Then f is constant, i.e., there exists a real constant $c \in \mathbb{R}$ such that $f = T_c$.*

Proof. By hypothesis, for every $\varphi \in \mathcal{D}((a,b))$ one has $\langle f, \varphi' \rangle = 0$. Given a function $\psi \in \mathcal{D}((a,b))$, its primitive

$$\varphi(x) = \int_{-\infty}^{x} \psi(t) dt$$

is identically constant on the interval $[A, \infty)$, where $A < b$ is the sup of the support of ψ. Hence, φ is in $\mathcal{D}((a,b))$ if and only if

$$J(\psi) := \int_{-\infty}^{+\infty} \psi(t) dt = 0.$$

Now fix a function $\tau_0 \in \mathcal{D}((a,b))$ such that $J(\tau_0) = 1$ and given $\phi \in \mathcal{D}((a,b))$ set $\psi = \phi - J(\phi)\tau_0$. Then $J(\psi) = 0$ and so $\psi = \varphi'$ for some $\varphi \in \mathcal{D}((a,b))$. Therefore, $\langle f, \psi \rangle = 0$ and $\langle f, \phi \rangle = \langle f, \tau_0 \rangle J(\phi) = const J(\phi)$ for every $\phi \in \mathcal{D}((a,b))$, which proves the theorem. $\square$

Corollary 9.2. *Let $f \in \mathcal{D}'((a,b))$ and $f' \in C((a,b))$. Then f is a regular distribution and $f \in C^1((a,b))$.*

Proof. The continuous function f' admits a primitive $\tilde{f}$ of class $C^1((a,b))$. Then $(f - \tilde{f})' = 0$ in $\mathcal{D}'((a,b))$ and Theorem 9.1 can be applied. $\square$

We now extend these results to distributions in several variables.

Theorem 9.3. *Let Ω' be a domain in $\mathbb{R}^{n-1}$ and $I = (a,b)$ be an interval in $\mathbb{R}$. Assume that a distribution $f \in \mathcal{D}'(\Omega' \times I)$ satisfies*

$$\frac{\partial f}{\partial x_n} = 0$$

in $\mathcal{D}'(\Omega' \times I)$. Then there exists a distribution $\tilde{f} \in \mathcal{D}'(\Omega')$ such that for every $\varphi \in \mathcal{D}(\Omega' \times I)$

$$\langle f, \varphi \rangle = \int_{\mathbb{R}} \langle \tilde{f}(x'), \varphi(x', x_n) \rangle dx_n,$$

where $x' = (x_1, \ldots, x_{n-1})$. In this sense the distribution f is independent of the variable x_n.

Proof. Fix a function $\tau_0 \in \mathcal{D}(I)$ such that $\int_{\mathbb{R}} \tau_0 dt = 1$. We lift every $\phi \in \mathcal{D}(\Omega')$ to a function $\tilde{\phi} \in \mathcal{D}(\Omega' \times I)$ by setting $\tilde{\phi}(x', x_n) = \phi(x')\tau_0(x_n)$. This allows us to define a distribution $\tilde{f} \in \mathcal{D}'(\Omega')$ by setting $\langle \tilde{f}, \phi \rangle = \langle f, \tilde{\phi} \rangle$, $\phi \in \mathcal{D}(\Omega')$.

Given $\psi \in \mathcal{D}(\Omega' \times I)$ put

$$J(\psi)(x') = \int_{\mathbb{R}} \psi(x', x_n) dx_n.$$

Similarly to the proof of Theorem 9.1 for every $\psi \in \mathcal{D}(\Omega' \times I)$ there exists a function $\varphi \in \mathcal{D}(\Omega' \times I)$ such that
$$\psi(x) - J(\psi)(x')\tau_0(x_n) = \frac{\partial \varphi(x)}{\partial x_n}.$$
Then by the assumptions of the theorem, $\langle f, \frac{\partial \varphi(x)}{\partial x_n}\rangle = 0$, and by the definition of the distribution $\tilde f$ we have
$$\langle f, \psi\rangle = \langle f, J(\psi)(x')\tau_0(x_n)\rangle = \langle \tilde f, J(\psi)\rangle = \langle \tilde f, \int_{\mathbb{R}} \psi(x', x_n)dx_n\rangle.$$
It remains to show that
$$\langle \tilde f, \int_{\mathbb{R}} \psi(x', x_n)dx_n\rangle = \int_{\mathbb{R}} \langle \tilde f(x'), \psi(x', x_n)\rangle dx_n.$$
Fix $\psi \in \mathcal{D}(\Omega' \times I)$ and consider the functions $F_1(x_n) = \langle \tilde f(x'), \int_{-\infty}^{x_n} \psi(x', t)dt\rangle$ and $F_2(x_n) = \int_{-\infty}^{x_n} \langle \tilde f(x'), \psi(x', t)\rangle dt$. Then it follows from Theorem 8.7 that $F_1' = F_2'$. Since $\lim_{x_n \to -\infty} F_j = 0$, we obtain $F_1 \equiv F_2$. This concludes the proof. ☐

Corollary 9.4. *Let $f \in \mathcal{D}(\Omega)$ satisfy $\frac{\partial f}{\partial x_j} = 0$, $j = 1, ..., n$. Then f is constant.*

Finally we establish a weak, but useful analogue of Corollary 9.2.

Theorem 9.5. *Let f and g be continuous functions in a domain $\Omega \subset \mathbb{R}^n$. Suppose that*
$$\frac{\partial T_f}{\partial x_n} = T_g.$$
Then the usual partial derivative $\frac{\partial f}{\partial x_n}$ exists at every point $x \in \Omega$ and is equal to $g(x)$.

Proof. The statement in local so without loss of generality we assume that $\Omega = \Omega' \times I$ in the notation of Theorem 9.3. Fix a point $c \in I$ and set
$$v(x) = \int_c^{x_n} g(x', t)dt.$$
Then $\frac{\partial(f-v)}{\partial x_n} = 0$ in $\mathcal{D}'(\Omega' \times I)$ and Theorem 9.3 gives the existence of a distribution $\tilde f \in \mathcal{D}'(\Omega')$ such that $f - v = \tilde f$. Furthermore, since $f - v$ is continuous, it follows from the construction of $\tilde f$ in the proof of Theorem 9.3 that $\tilde f$ is a continuous function in x' (defining a regular distribution). Then the function $f(x) = v(x) + \tilde f(x')$ admits a partial derivative in x_n which coincides with g. This proves the theorem. ☐

9.3. **Support of a distribution. Distributions with compact support.** Let $f \in \mathcal{D}'(\Omega')$, and $\Omega \subset \Omega'$ be a subdomain. By the restriction of f to Ω we mean a distribution $f|\Omega$ acting by
$$\langle f|\Omega, \varphi\rangle := \langle f, \varphi|\Omega\rangle,\ \varphi \in \mathcal{D}(\Omega) \subset \mathcal{D}(\Omega').$$
We say that a distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ vanishes on an open subset $U \subset \mathbb{R}^n$ if $\langle f, \varphi\rangle = 0$ for any $\varphi \in \mathcal{D}(U)$, i.e., its restriction to U vanishes identically. We express this as $f|U \equiv 0$.

Definition 9.6. *The support* $\operatorname{supp} f$ *of a distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ is the subset of $\mathbb{R}^n$ with the following property: $x \in \operatorname{supp} f$ if and only if for every neighbourhood U of x there exists $\phi \in \mathcal{D}(U)$ (and so $\operatorname{supp} \phi \subset U$) such that $\langle f, \phi\rangle \neq 0$, i.e., f does not vanish identically in any neighbourhood of x.*

It follows from the definition of $\operatorname{supp} f$ that it is a closed subset of $\mathbb{R}^n$, and so its complement is an open (but not necessarily connected) subset of $\mathbb{R}^n$. Indeed, the set $\mathbb{R}^n \backslash \operatorname{supp} f$ is formed by all points x such that f vanishes identically in some neighbourhood of x and so it is clearly open.

Proposition 9.7. *Let X be an open subset of $\mathbb{R}^n$ such that $f \in \mathcal{D}'(\mathbb{R}^n)$ vanished identically in a neighbourhood of every point of X. Then $f|_X \equiv 0$.*

Proof. Given point $x \in X$ there exists a neighbourhood U_α such that $f|_{U_\alpha} \equiv 0$. Let $\phi \in \mathcal{D}(X)$. Consider a neighborhood U of $\operatorname{supp} \phi$ such that the closure $\overline{U}$ is a compact subset of X. Let (η_γ) be a partition of unity subordinated to a finite sub-covering (U_α) of $\overline{U}$ (see Section 7). Then $\langle f, \phi \rangle = \sum_\gamma \langle f, \eta_\gamma \phi \rangle = 0$ since every $\eta_\gamma \phi \in \mathcal{D}(U_\alpha)$ for some α. □

Example 9.5. If f is a regular distribution defined by a function $f \in C(\mathbb{R}^n)$ then its support in the sense of distributions coincides with the support in the usual sense since f vanishes on an open set U as a distribution if and only if it vanishes as a usual function. ◇

Example 9.6. $\operatorname{supp} \delta(x) = \{0\}$. ◇

A remarkable property of distributions with a compact support in $\mathbb{R}^n$ is that one can extend them as linear continuous functionals defined on the space $C^\infty(\mathbb{R}^n)$. Let $f \in \mathcal{D}'(\mathbb{R}^n)$ have a compact support $\operatorname{supp} f = K$ in $\mathbb{R}^n$. Fix a function $\eta \in C_0^\infty(\mathbb{R}^n)$ such that $\eta(x) = 1$ in a neighbourhood of K. Then for every $\psi \in C^\infty(\mathbb{R}^n)$ the function $\eta\psi$ is in $\mathcal{D}(\mathbb{R}^n)$ and we set

$$\langle f, \psi \rangle := \langle f, \eta\psi \rangle, \tag{1}$$

since the right-hand side is well-defined. This definition is independent of the choice of η. Indeed, let $\eta' \in C_0^\infty(\mathbb{R}^n)$ be another function vanishing in a neighbourhood of K. Then $\eta - \eta'$ vanishes in a neighbourhood of K and for any $\psi \in C^\infty(\mathbb{R}^n)$ the support of the function $(\eta - \eta')\psi \in \mathcal{D}(\mathbb{R}^n)$ is contained in $\mathbb{R}^n \backslash K$. By Definition 9.6 we have

$$\langle f, \eta\psi \rangle - \langle f, \eta'\psi \rangle = \langle f, (\eta - \eta')\psi \rangle = 0.$$

Hence, (1) is independent of the choice of η. The defined above extension of f (still denoted by f) is, of course, a linear continuous functional on $C^\infty(\mathbb{R}^n)$. Indeed, let a sequence ψ^k converge to ψ in $C^\infty(\mathbb{R}^n)$, i.e., ψ^k converges to ψ together with all derivatives uniformly on every compact subset of $\mathbb{R}^n$. Then $\eta\psi^k$ converges to $\eta\psi$ in $\mathcal{D}(\mathbb{R}^n)$ and

$$\langle f, \psi^k \rangle = \langle f, \eta\psi^k \rangle \longrightarrow \langle f, \eta\psi \rangle = \langle f, \psi \rangle.$$

Example 9.7. For every $\psi \in C^\infty(\mathbb{R}^n)$ we have

$$\langle \delta(x), \psi \rangle = \langle \delta(x), \eta\psi \rangle = \eta(0)\psi(0) = \psi(0)$$

since $\eta = 1$ in a neighbourhood of $\operatorname{supp} \delta(x) = \{0\}$. ◇